Holomorphic Sectional Curvature of Projectivized Vector Bundles over Compact Complex Manifolds

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In geometry, having positive curvature on a manifold yields beautiful and interesting structural consequences.

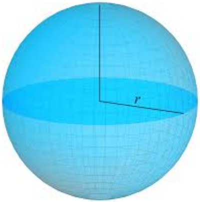
For instance, in Riemannian geometry, we have...

Theorem (Bonnet-Myers)

Let (M, g) be an n-dimensional complete Riemannian manifold. If $\exists\ k \in \mathbb{R}^+$ such that

$$Ric(M) \geq (n-1)k > 0$$

then the diameter of M is no greater than $\frac{\pi}{\sqrt{k}}$.



In the world of complex geometry...

Manifolds of Positive Curvature	*vs.*	Manifolds of Negative Curvature

In negative curvature:

- Plenty of known examples of, and many known results about, negatively-curved manifolds.

In positive curvature:

- Few known examples of manifolds with positive curvature.
- Tend to be fewer known results about positively-curved manifolds.

Many difficulties arise when dealing with positive curvature..

We will focus on the **holomorphic sectional curvature** of compact complex manifolds.

It has signficant relationships to various notions in algebraic geometry.

For example:

- (Heier-Lu-Wong '10, Wu-Yau '16, Tosatti-Yang '17)
  M has negative holomorphic sectional curvature $\implies K_M$ is ample.
- (Heier-Lu-Wong '16)
  M has semi-negative holomorphic sectional curvature $\implies$ Lower bounds for the nef dimension and Kodaira dimension of M
- (Yang '18)
  M compact Kähler manifold with positive holomorphic sectional curvature $\implies M$ projective and rationally connected

- There are few examples of compact complex manifolds with positive holomorphic sectional curvature (as well as positively-pinched).
- Many difficulties arise when dealing with holomorphic sectional curvature in the positive case.

  For Example:

  The Decreasing Property of the Holomorphic Sectional Curvature on Submanifolds: Let M be a Hermitian manifold and let N be a complex submanifold of M. Then the holomorphic sectional curvature of N does not exceed that of M.

Goal:

1. Find metrics of positive holomorphic sectional curvature on complex manifolds.
2. Investigate any structural consequences brought about by metrics of positive holomorphic sectional curvature.

In this talk:

- Existence [pinched] metrics of positive holomorphic sectional curvature on certain fibrations $\pi : P \to M$ where

  $P =$ projectivized vector bundle
  $M =$ compact complex manifold of positive holomorphic sectional curvature.
- Curvature pinching results for projectivized rank 2 vector bundles over $\mathbb{CP}^1$.

Definition

Let M be an n-dimensional complex manifold and let $p \in M$. A Hermitian metric on M is a positive definite Hermitian inner product

$$g_p : T'_pM \otimes \overline{T'_pM} \to \mathbb{C}$$

which varies smoothly for each $p \in M$.

Let $\{dz_1, ..., dz_n\}$ be the dual basis of $\{\frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_n}\}$.

Locally, the Hermitian metric can be written as

$$g = \sum_{i,j=1}^{n} g_{i\bar{j}} dz_i \otimes d\bar{z}_j$$

where $\left(g_{i\bar{j}}\right)$ is an $n \times n$ positive definite Hermitian matrix of smooth functions.

The metric g can be decomposed into two parts:

1. The Real Part, denoted by $Re(g)$
2. The Imaginary Part, denoted by $Im(g)$.

$Re(g)$ gives an ordinary inner product called the induced Riemannian metric of g.

$Im(g)$ represents an alternating $\mathbb{R}$-differential 2-form.

Let $\omega := -\frac{1}{2} Im(g)$.

Definition

The $(1,1)$-form ω is called the associated $(1,1)$-form of g.

Definition

The Hermitian metric g is called Kähler if $d\omega = 0$, where d is the exterior derivative $d = \partial + \bar{\partial}$.

There are several equivalences for a metric being Kähler.

One of them being:

A metric g is Kähler	$\Longleftrightarrow$	For any $p \in M$, $\exists$ holomorphic coordinates $(z_1, ..., z_n)$ near p such that $g_{i\bar{j}}(p) = \delta_{ij}$ and $(dg_{i\bar{j}})(p) = 0$.

Such coordinates are called normal coordinates.

Definition

Let $X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial z_i}$ be a non-zero complex tangent vector at $p \in M$. Then the holomorphic sectional curvature in the direction of X is

$$
\begin{aligned}
K(X) &= \left(2 \sum_{i,j,k,l=1}^{n} R_{i\bar{j}k\bar{l}}(p) X_i \bar{X}_j X_k \bar{X}_l\right) \Big/ \left(\sum_{i,j,k,l=1}^{n} g_{i\bar{j}} g_{k\bar{l}} X_i \bar{X}_j X_k \bar{X}_l\right) \\
&= \frac{2R_{X\bar{X}X\bar{X}}}{|X|^4}
\end{aligned}
$$

where

$$
R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + \sum_{p,q=1}^{n} g^{q\bar{p}} \frac{\partial g_{i\bar{p}}}{\partial z_k} \frac{\partial g_{q\bar{j}}}{\partial \bar{z}_l}
$$

*For a Kähler manifold: K is the Riemannian sectional curvature of the holomorphic planes in the tangent space of the manifold.

Examples:

1. $\mathbb{CP}^n$, together with the Fubini-Study Metric
$$\omega = \frac{\sqrt{-1}}{2}\partial\bar{\partial}\log|w|^2,$$
has positive holomorphic sectional curvature equal to 4.

2. $\mathbb{C}^n$ has holomorphic sectional curvature equal to 0.

3. $\mathbb{B}^n$ has negative holomorphic sectional curvature.

Definition

Let M be a **compact** Hermitian manifold with holomorphic sectional curvature $K(X)$. Let $c \in (0, 1]$. We say that the holomorphic sectional curvature is c-pinched if

$$\frac{\min_X K(X)}{\max_X K(X)} = c \ \ (\leq 1)$$

where the maximum and minimum are taken over all (unit) tangent vectors across M.

c = "pinching constant"

Pinching constants can help determine some global properties of the manifold.

- (Sekiwaga-Sato '85)
  Nearly Kähler manifolds with $c > \frac{2}{5}$ are isometric to the 6-sphere of constant curvature $\frac{1}{30}$*(scalar curvature)

- (Bracci-Gaussier-Zimmer '18)
  Existence of a negatively-pinched Kähler metric on a domain in complex Euclidean space restricts the geometry of its boundary

This work was partially inspired by the following result by C.-K. Cheung in negative curvature:

Theorem (Cheung, 1989)

Let $\pi : X \to Y$ be a holomorphic map from a compact complex manifold X into a complex manifold Y which has a Hermitian metric of negative holomorphic sectional curvature. Assume:

- *π is of maximal rank everywhere.*
- *There exists a smooth family of Hermitian metrics on the fibers, which all have negative holomorphic sectional curvature.*

There there exists a Hermitian metric on X with negative holomorphic sectional curvature everywhere.

We naturally asked:

Does the result of Cheung still hold true for metrics of positive holomorphic sectional curvature?

As a primary stepping stone, we considered *projectivized* vector bundles thanks to:

Theorem (Hitchin, 1975)

For $n \in \mathbb{Z}^{\geq 0}$, the n^{th} Hirzebruch surfaces [projectivized rank 2 vector bundles over $\mathbb{CP}^1$] admits Kähler metrics of positive holomorphic sectional curvature.

S.-T. Yau posed the following open question in *Riemannian* geometry:

Do all vector bundles over a manifold with positive curvature admit a complete metric with nonnegative sectional curvature?

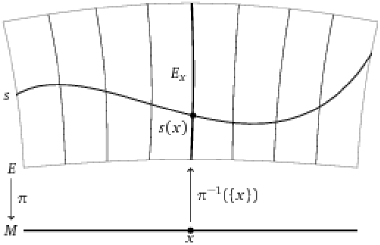
Transplant Yau's question to the complex setting and projectivize the vector bundle

$$\text{“nonnegative curvature”} \longleftrightarrow \text{“positive curvature”}.$$

We arrive at an affirmative answer, which serves as a generalization of Hitchin's theorem:

Theorem (A.-Heier-Zheng, 2018)

Let M be an n-dimensional compact Kähler manifold. Let E be holomorphic vector bundle over M and let $\pi : P = \mathbb{P}(E) \to M$ be the projectivization of E. If M has positive holomorphic sectional curvature, then P admits a Kähler metric with positive holomorphic sectional curvature.



Let (M, g) be an n-dimensional compact Kähler manifold with positive holomorphic sectional curvature. Let ω_g be the associated $(1,1)$-form of g.

Let E be a rank $(r+1)$-vector bundle on M, with arbitrary Hermitian metric h.

Let $(x, [v])$ be a moving point on P.

The metrics g and h induce a closed associated $(1,1)$-form on P:

$$\omega_G = \pi^*(\omega_g) + s\sqrt{-1}\partial\bar{\partial}\log h_{v\bar{v}}$$

which is the associated $(1,1)$-form on $G := G_s$.

For s sufficiently small, ω_G is positive definite everywhere and thus is a Kähler metric on P.

Using normal coordinates, we prove that for s sufficiently small (depending on g and h) G has positive holomorphic sectional curvature.

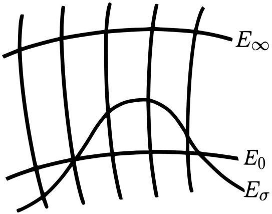
Natural Question: What about pinching constants?

- First investigate pinching constants of projectivized rank 2 vector bundles where the base manifold is $\mathbb{CP}^1$.

Definition

The n^{th} Hirzebruch Surface, $n \in \mathbb{Z}^{\geq 0}$ is defined to be

$$\mathbb{F}_n := \mathbb{P}\left(\mathcal{O}_{\mathbb{CP}^1}(n) \oplus \mathcal{O}_{\mathbb{CP}^1}\right) \to \mathbb{CP}^1, \quad n \in \mathbb{Z}^{\geq 0}$$



The form of the Kähler metric that Hitchin used to prove $\mathbb{F}_n$ has positive holomorphic sectional curvature is:

$$\varphi_s = \sqrt{-1}\partial\bar{\partial}\left(\log(1 + z_1\bar{z}_1) + s\log((1 + z_1\bar{z}_1)^n + z_2\bar{z}_2)\right)$$

where (z_1, z_2) are inhomogeneous coordinates on $\mathbb{F}_n$, and $s \in \mathbb{R}^+$ is chosen small enough such that φ_s is positive definite.

- $\mathbb{F}_n$ is compact, but Hitchin's proof did not yield any pinching constants.

As a result, we have the following pinching result:

Theorem (A.-Chaturvedi-Heier, 2015)

Let $\mathbb{F}_n$, $n \in \{1, 2, 3, \dots\}$, be the n-th Hirzebruch surface. Then there exists a Kähler metric on $\mathbb{F}_n$ whose holomorphic sectional curvature is $\frac{1}{(1+2n)^2}$-pinched.

Theorem (A.-Chaturvedi-Heier, 2015)

Let M and N be Hermitian manifolds whose positive holomorpic sectional curvatures are c_M- and c_N-pinched, respectively. Then the holomorphic sectional curvature of the product metric on $M \times N$ is also positive, and is $\frac{c_M c_N}{c_M + c_N}$-pinched.

In Riemannian Geometry:

The Hopf Conjecture: The product of two real 2-spheres DOES NOT carry a Riemannian metric of positive sectional curvature.

The result just shows that holomorphic sectional curvature is more well-behaved than (Riemannian) sectional curvature.

Questions for Future Work:

1. What would be a pinching constant for a general projectivized rank k vector bundle over $\mathbb{CP}^1$?

2. What would be a pinching constant of $\mathbb{P}(E)$ over M, where M is an *arbitrary* compact complex manifold of positive holomorphic sectional curvature?
   - (Yang-Zheng '16) Let

     $$M_{n,k} = \mathbb{P}\left(\mathcal{O}_{\mathbb{CP}^{n-1}}(k) \oplus \mathcal{O}_{\mathbb{CP}^{n-1}}\right)$$

     The (local) pinching constant of the holomorphic sectional curvature of any $U(n)$ invariant Kähler metric on $M_{n,k}$ is bounded above by $\frac{1}{k^2}$.

Thank you.