The Kakeya needle problem for rectifiable sets

with Alan Chang

The Kakeya needle problem (geometric version)

$E \subset \mathbb{R}^2$ has the Kakeya property if it can be moved continuously between two different positions covering arbitrary small area.

$E \subset \mathbb{R}^2$ has the strong Kakeya property if it can be moved between any two positions covering arbitrary small area.

- E.g. circles and lines have the Kakeya property, but not the strong Kakeya property.
- Any subset of a null set of parallel lines or concentric circles have the Kakeya property.
- Finitely many parallel line segments have the strong Kakeya property (Davies).
- A short enough circular arc has the strong Kakeya property (Héra, Laczkovich).

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Theorem (C., Héra, Laczkovich)

- If E is closed and connected, and admits the Kakeya property, then $E \subset$ line or circle.
- If E is closed and admits the Kakeya property, then the non-trivial connected components of E are covered by a null set of parallel lines or concentric circles.

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The Kakeya needle problem (analyst's version)

Joint work with Chang

- A (full) line can be moved continuously to any other position, covering only zero area, provided that at each time moment we are allowed to delete 1 point.
  (A continuous Nikodym set.)
- A circle can be moved continuously to any other position, covering only zero area, provided that at each time moment we are allowed to delete 2 points.
  (Every circular arc shorter than a half-circle has the strong Kakeya property.)

Open problem. What happens for circular arcs longer than half-circle?

Using not only isometries but also similarities, a circle can be moved continuously to any other position, covering only zero area, provided that at each time moment we are allowed to delete 1 point.

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Nikodym sets for circles

There are NO sets in $\mathbb{R}^2$ which have measure zero and contain a circle centered at every point.

- $\mathbb{R}^n$, $n \geq 3$: Stein.
- $n = 2$: Bourgain, Marstrand.

The non-existence results concern placing a circle around every point of $\mathbb{R}^2$. For our Nikodym result, we place a circle through every point of $\mathbb{R}^2$. With this change such a construction is now possible.

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- A parabola can be moved, using only translations, into any other shifted position, if we are allowed to delete 1 point. It can be also rotated into any other rotated position if we are allowed to delete 1 point.
- The graph of every convex function can be rotated, deleting only 1 point.
- The graph of every strictly convex function can be shifted, deleting only 1 point.
- The graph of $x \to x^3$ can be moved into any other position, deleting only 2 points.
- Etc, etc. As it turns out, neither the topological nor the algebraic structure of the curve plays any role. Our main result holds for every rectifiable curve, and depends on its tangential properties.

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What about dimension?

Let Γ be a rectifiable curve. Is it true that if a set contains a rotated copy of Γ in each direction, then it has full dimension?

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- Surprisingly, no.
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The Venetian blind idea for translations

Let E be a rectifiable set, and θ_x the tangent direction at $x \in E$. Fix $\delta > 0$, and fix a direction θ. Consider those points $x \in E$ for which $|\theta_x - \theta| \lesssim \delta$.

This set can be covered by countably many Lipschitz curves Γ_i, each Γ_i is the graph of a Lipschitz function f_i with Lipschitz constant $\lesssim \delta$ in the $(\theta, \theta^\perp)$ coordinate system. How much area we cover if we shift $E \cap \Gamma_i$ in the θ direction by a vector v?

$$\begin{aligned} \text{Area} &\leq |v| \int \#\{x \in \mathbb{R} : f(x) = t, (x, f(x)) \in E \cap \Gamma_i\}\, dt \\ &\lesssim \delta |v| \mathcal{H}^1(E \cap \Gamma_i). \end{aligned}$$

Summing over i, we obtain $\delta |v| \mathcal{H}^1(E)$.

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Main result: translations.

Suppose that E is a rectifiable set with a nice enough tangent field.

Remark. We can always through away an $\mathcal{H}^1$-null set and find a "nice enough" tangent field.

Theorem. *E can be moved using only translations into any other shifted position, covering zero area, provided that at each time moment we are allowed to delete points of a given tangent direction.*

Some examples:

- A circle can be moved if at each time moment we are allowed to delete two diametrically opposite points.
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Main result: rotation.

Same assumptions.

Theorem. E can be moved into any other position, covering zero area, provided that at each time moment we are allowed to delete points whose normal line goes through a given point.

Examples:

- Circle: two diametrically opposite points.
- Line: only one point.

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