# Quasi-symmetric & biLipschitz maps of boundaries of negatively curved homogeneous spaces

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# Maps between metric spaces

$$f : (X, d) \longrightarrow (Y, d)$$

- Quasi-symmetric

$$\frac{d(f(x), f(y))}{d(f(y), f(z))} \leq \eta\left(\frac{d(x,y)}{d(y,z)}\right)$$

- Bi-Lipschitz

$$a\, d(x,y) \leq d(f(x), f(y)) \leq b\, d(x,y)$$

- Similarity

$$d(f(x), f(y)) = a\, d(x,y)$$

## Questions

① For which metric spaces are quasi-symmetric maps biLipschitz?

② Under which conditions is a group of biLipschitz / quasi-symmetric maps (a conjugate of) a group of similarities?

## The metric spaces

$$(N, d_A)$$

nilpotent Lie gp admitting family of expanding automorphisms

$\delta_t : N \to N$

parabolic visual metric

left invariant metric such that

$$d_A(\delta_t(n_1), \delta_t(n_2)) = e^t d_A(n_1, n_2)$$

$A : \mathfrak{n} \to \mathfrak{n}$ derivation on Lie algebra with all eigenvalues with positive real part

$\updownarrow$ diffeo

$\delta_t = e^{tA} : N \to N$

e.g. $N = \mathbb{R}^2 = \{(x,y)\}$ $\quad A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

$$d_A\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = \max\left\{ |x_1 - x_2|, |y_1 - y_2|^{\frac{1}{2}} \right\}$$

e.g. $N =$ Heisenberg group

$= \{(x, y, z)\}$

$$A = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix}$$

$$\max\left\{ |\delta x|, |\delta y|, |\delta z|^{\frac{1}{2}} \right\} \qquad \max\left\{ |\delta x|, |\delta y|^{\frac{1}{2}}, |\delta z|^{\frac{1}{3}} \right\}$$

# Connections to Geometric Group Theory

$$H_A := N \rtimes_A \mathbb{R}$$

is negatively curved as long as eigenvalues of $A$ have positive real part.

e.g. $N = \mathbb{R}^2 \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$H_A = \mathbb{R}^2 \rtimes \mathbb{R} \simeq \mathbb{H}^3$$

e.g. $N = \text{Heisenberg} \quad A = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 2 \end{pmatrix}$

$$H_A = N \rtimes \mathbb{R} \simeq \mathbb{C}\mathbb{H}^2$$

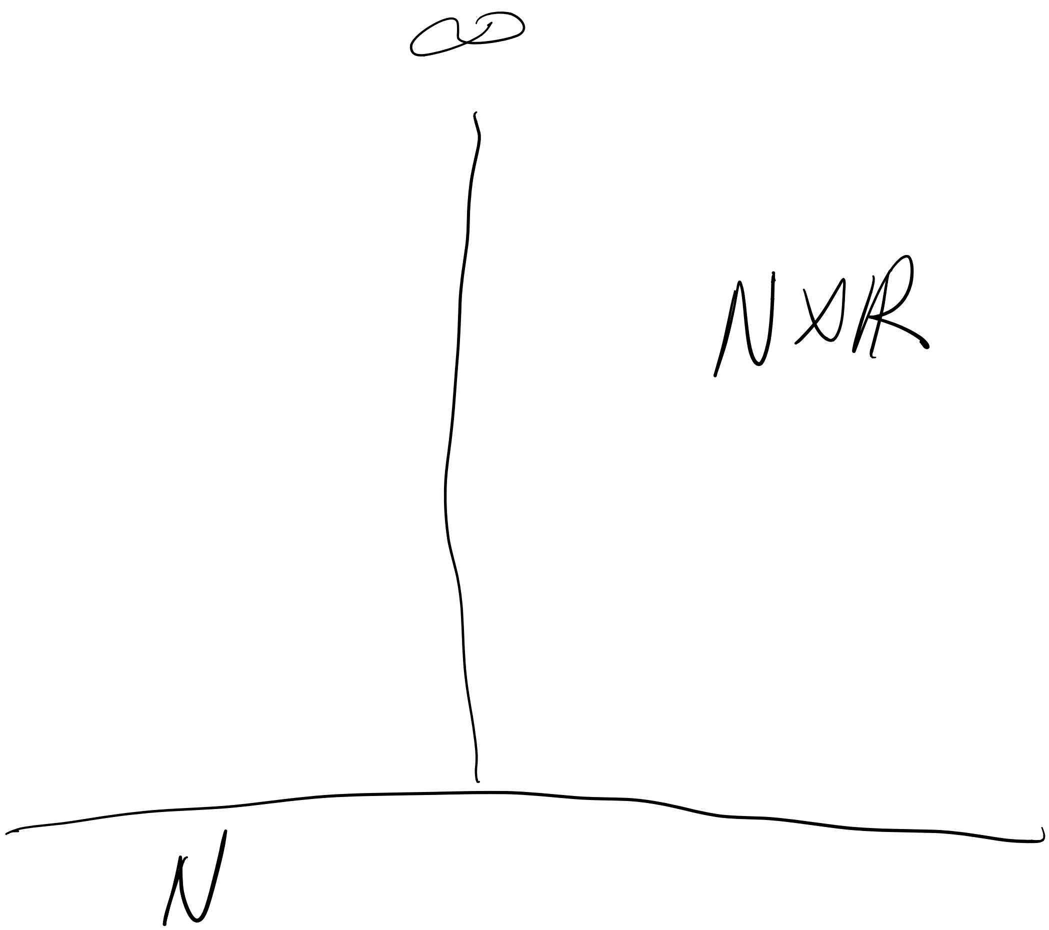
In general $H_A$ is negatively curved homogeneous space.

# Boundaries

"vertical" geodesics

e.g. $\partial \mathbb{H}^3 = \mathbb{R}^2 \cup \{\infty\}$

$$\partial H_A = (N, d_A) \cup \{\infty\}$$



# Dictionary

| $\partial H_A$ | $H_A$ |
|---|---|
| quasi-symmetric maps | quasi-isometries |
| bilipschitz maps | height respecting quasi-isometries |
| similarities | isometries |

<u>Conjecture</u> ① All q.s maps of $(N, d_A)$ are biLip unless $H_A$ isometric to symmetric space.

(Xie)

<u>Conjecture</u> ② Under certain conditions:

ⓐ uniformity

ⓑ co-compact action on distinct pairs

ⓒ amenability

A group of bi-Lipschitz maps of $(N, d_A)$ is the conjugate of a group of similarities of $(N, d_{A'})$

(Xie - Dymarz)

# Notes

- Tukia: case of quasi-conformal maps of $S^n = \mathbb{R}^n \cup \{\infty\}$
  $= \partial \mathbb{H}^{n+1}$

- Chow: case of quasi-conformal maps of $\partial \mathbb{C}\mathbb{H}^n$

- Pansu: boundaries of remaining symmetric spaces, no conjugation needed.

- Xie: case of quasi-conformal maps of $(N, d_A)$ Carnot groups.

# Some Theorems (Xie - Dymarz)

- $(\mathbb{R}^n, d_A)$    $A$ diagonal (izable)
  $G$ amenable, uniform group of BiLipschitz maps then it is a conjugate of similarity group of $(\mathbb{R}^n, d_A)$

- $(\text{Heis}, d_A)$    $A = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix}$
  same conclusion

- In progress: $(N, d_A)$
  $A$ diagonal (izable)