A C^m Whitney Extension Theorem for Horizontal Curves in the Heisenberg Group

Gareth Speight

University of Cincinnati

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Topics at the Interface of Analysis and Geometry

Jets

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Definition

A **jet of order** m **on** K is a collection $F = (F^k)_{|k| \leq m}$ of continuous functions on K.
Here $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$ is a multi index and $|k| = k_1 + \ldots + k_n$.

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For each C^m function $f \colon \mathbb{R}^n \to \mathbb{R}$ there is an associated jet on K:

$$f \mapsto \left(\left. \frac{\partial^{|k|} f}{\partial x^k} \right|_K \right)_{|k| \leq m}.$$

Whitney Jets

Given a jet $(F^k)_{|k|\leq m}$, we denote for $|k| \leq m$:

$$(R_x^m F)^k(y) = F^k(y) - \sum_{|l|\leq m-|k|} \frac{F^{k+l}(x)}{l!}(y-x)^l.$$

Here $(y-x)^l = (y_1 - x_1)^{l_1} \cdots (y_n - x_n)^{l_n}$.

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$(F^k)_{|k|\leq m}$ is a **Whitney field of class** C^m **on** K if for all $|k| \leq m$:

$$\frac{(R_x^m F)^k(y)}{|x-y|^{m-|k|}} \to 0$$

uniformly as $|x-y| \to 0$ with $x, y \in K$.

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The jet associated to a C^m function is always a Whitney field of class C^m on any compact set K.

Whitney Extension Theorem

Theorem (Whitney)

Let $(F^k)_{|k|\leq m}$ be a Whitney field of class C^m on K. Then there exists a C^m map $f : \mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial^{|k|} f}{\partial x^k}\Big|_K = F^k \qquad \text{for } |k| \leq m.$$

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Corollary (Lusin Approximation of Curves)

Suppose $\gamma\colon [a,b] \to \mathbb{R}^n$ *is absolutely continuous and* $\varepsilon > 0$*.*
Then there exists a C^1 *curve* $\Gamma\colon [a,b] \to \mathbb{R}^n$ *such that*

$$\mathcal{L}^1\{t \in [a,b] \colon \Gamma(t) \neq \gamma(t) \text{ or } \Gamma'(t) \neq \gamma'(t)\} < \varepsilon.$$

Heisenberg Group

Definition

The **first Heisenberg group** $\mathbb{H}^1$ is $\mathbb{R}^3$ equipped with group law:

$$(x, y, t)(x', y', t') = (x + x', y + y', t + t' - 2(xy' - yx')).$$

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Left invariant **horizontal vector fields** on $\mathbb{H}^1$ are defined by:

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- **Dilations** are defined by $\delta_r(x, y, t) = (rx, ry, r^2 t)$. They satisfy

$$\delta_r(ab) = \delta_r(a)\delta_r(b)$$

and

$$\mathcal{L}^3(\delta_r(A)) = r^4 \mathcal{L}^3(A).$$

Horizontal Curves

Definition

An absolutely continuous curve $\gamma\colon [a,b] \to \mathbb{H}^1$ is **horizontal** if there exists $h\colon [a,b] \to \mathbb{R}^2$ such that for almost every t:

$$\gamma'(t) = h_1(t)X(\gamma(t)) + h_2(t)Y(\gamma(t)).$$

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The **horizontal length** of such a curve is defined by:

$$L(\gamma) = \int_a^b |h|.$$

Any two points can be connected by a horizontal curve!

Horizontal Lift

Lemma (Horizontal Lift)

An absolutely continuous curve $\gamma\colon [a,b] \to \mathbb{H}^1$ *is horizontal if and only if*

$$\gamma_3(t) = \gamma_3(a) + 2\int_a^t (\gamma_1'\gamma_2 - \gamma_2'\gamma_1)$$

for every $t \in [a,b]$.

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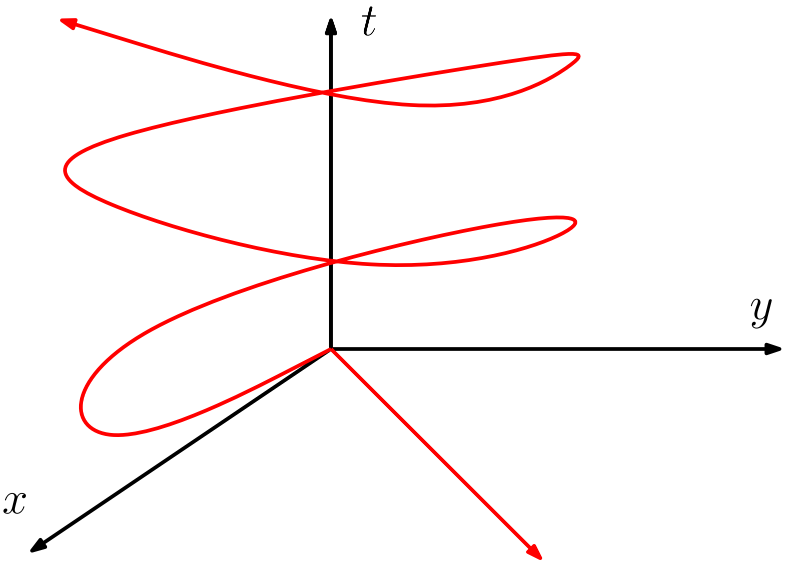
Lemma (Height-Area Interpretation)

Suppose $\sigma\colon [a,b] \to \mathbb{R}^2$ *is a smooth curve with* $\sigma(a) = 0$. *Let* A_σ *denote the signed area of the region enclosed by* σ *and the straight line* $[0, \sigma(b)]$. *Then*

$$A_\sigma = \frac{1}{2}\int_a^b (\sigma_1\sigma_2' - \sigma_2\sigma_1').$$

Horizontal Curves



Whitney Extension for C^1 Horizontal Curves in $\mathbb{H}^1$

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Theorem (Zimmerman)

Suppose (f, f'), (g, g'), (h, h') are Whitney fields of class C^1 on K. Then there exists a C^1 horizontal curve $\Gamma \colon \mathbb{R} \to \mathbb{H}^1$ such that $\Gamma|_K = (f, g, h)$ and $\Gamma'|_K = (f', g', h')$ if and only if

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and

$$\frac{|h(b) - h(a) - 2(f(b)g(a) - f(a)g(b))|}{|b-a|^2} \to 0 \text{ as } |b-a| \to 0 \text{ with } a, b \in K.$$

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Theorem (Speight)

Let $\gamma\colon [0,1] \to \mathbb{H}^1$ and $\varepsilon > 0$ be an absolutely continuous horizontal curve. Then there is a C^1 horizontal curve $\Gamma\colon [0,1] \to \mathbb{H}^1$ such that:

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What about higher regularity than C^1?

Area Discrepency and Velocity

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For $a, b \in K$ define

$$A(a,b) = H(b) - H(a) - 2\int_a^b (T_aF)'(T_aG) - (T_aG)'(T_aF) + 2F(a)(G(b) - T_aG(b)) - 2G(a)(F(b) - T_aF(b))$$

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and

$$V(a,b) = (b-a)^{2m} + (b-a)^m \int_a^b |(T_aF)'| + |(T_aG)'|.$$

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Theorem (Pinamonti, Speight, Zimmerman)

*Let $F, G, H\colon K \to \mathbb{R}$ be Whitney fields of class C^m on K.
Then (F, G, H) extends to a C^m horizontal curve from $\mathbb{R}$ into $\mathbb{H}^1$ if and only if both of the following conditions hold:*

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1. *for every $1 \leq k \leq m$ and $t \in K$ we have*

$$H^k(t) = \mathcal{P}^k(F^0(t), G^0(t), F^1(t), G^1(t), \cdots, F^k(t), G^k(t))$$

   where polynomials $\mathcal{P}^k$ come from differentiating the horizontality condition,

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Proof uses classical Whitney extension theorem and perturbations.

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Lemma

There exists a modulus of continuity $\beta \geq \alpha$ so that for each interval $[a_i, b_i]$, there exist C^∞ functions $\phi, \psi \colon [a_i, b_i] \to \mathbb{R}$ such that

1. $D^k\phi(a_i) = D^k\phi(b_i) = D^k\psi(a_i) = D^k\psi(b_i) = 0$ *for* $0 \leq k \leq m$.
2. $\max\{|D^k\phi|, |D^k\psi|\} \leq \beta(b_i - a_i)$ *for* $0 \leq k \leq m$ *on* $[a_i, b_i]$.
3. $H(b_i) - H(a_i) = 2\int_{a_i}^{b_i} (f+\phi)'(g+\psi) - (g+\psi)'(f+\phi)$.

Useful Inequalities

Theorem (Markov Inequality)

Let P be a polynomial of degree n and $a < b$. Then

$$\max_{[a,b]} |P'| \leq \frac{2n^2}{b-a} \max_{[a,b]} |P|$$

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Corollary

Let P be a polynomial of degree n and $a < b$. Then there exists a closed subinterval $I \subset [a,b]$ such that

- $l(I) \geq (b-a)/4n^2$
- $|P(x)| \geq \frac{1}{2} \max_{[a,b]} |P|$ *for all* $x \in I$.

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Thank you for listening!