Higher order rectifiability via Reifenberg theorems for sets and measures

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Parametrizing

History

- Reifenberg 1960: a "flat" set can be parametrized by a Hölder map.
  - The set is required to be flat and without holes: at every point and scale there's a plane close to the set and the set is close to the plane (official definition coming soon)

Parametrizing

History

- David-Kenig-Toro 2001: a “flat” set with small β numbers can be parametrized by a $C^{1,\alpha}$ map
  - The sets are “flat” with *vanishing constant*
- Kolasiński 2015: a “flat” set with small holes and small β numbers can be parametrized by a $C^{1,\alpha}$ map
  - Small holes $=$ size of β
  - Uses Menger-like curvatures

Parametrizing

History

- David-Toro 2012: a "flat" set with holes can be parametrized by a Hölder map
  - Moreover if we assume convergence of a Jones function then we can get a bi-Lipschitz parametrization
  - No control assumed on the size of the holes

Parametrizing

The first main theorem (vague statement)

- G. 2018: a “flat” set with holes can be parametrized by a $C^{1,\alpha}$ map if we assume a stronger convergence of the Jones function
  - Again, no control assumed on the size of the holes

Parametrizing

Definition of Reifenberg flat sets

Definition

Let $E \subseteq \mathbb{R}^n$ and let $\varepsilon > 0$. Define E to be *Reifenberg flat* if the following condition holds.

For $x \in E$, $0 < r \leq 10$ there is a d-plane $P_{x,r}$ such that

$$\begin{aligned} \operatorname{dist}(y, P_{x,r}) &\leq \varepsilon r, \qquad & y \in E \cap B(x,r), \\ \operatorname{dist}(y, E) &\leq \varepsilon r, \qquad & y \in P_{x,r} \cap B(x,r). \end{aligned}$$

Parametrizing

Definition of one-sided Reifenberg flat sets

Definition

Let $E \subseteq \mathbb{R}^n$ and let $\varepsilon > 0$. Define E to be one-sided *Reifenberg flat* if the following conditions hold.

(1) For $x \in E$, $0 < r \leq 10$ there is a d-plane $P_{x,r}$ such that

$$\mathrm{dist}(y, P_{x,r}) \leq \varepsilon r, \qquad y \in E \cap B(x,r),$$

$$\mathrm{dist}(y, E) \leq \varepsilon r, \qquad y \in P_{x,r} \cap B(x,r).$$

(2) Moreover we require some compatibility between the $P_{x,r}$'s:

$$d_{x,10^{-k}}(P_{x,10^{-k}}, P_{x,10^{-k+1}}) \leq \varepsilon, x \in E,$$

$$d_{x,10^{-k+2}}(P_{x,10^{-k}}, P_{y,10^{-k}}) \leq \varepsilon, x, y \in E, \ |x-y| \leq 10^{-k+2}$$

Parametrizing

Definition of β numbers

Definition

Let $E \subseteq \mathbb{R}^n$, $x \in \mathbb{R}^n$, and $r > 0$.

- β_∞:

$$\beta_\infty^E(x,r) = \inf_P \sup_{y \in E \cap B(x,r)} \frac{\operatorname{dist}(y,P)}{r}$$

  if $E \cap B(x,r) \neq \varnothing$, where the infimum is taken over all d-planes P, and $\beta_\infty^E(x,r) = 0$ if $E \cap B(x,r) = \varnothing$.

- β_p:

$$\beta_p^E(x,r) = \inf_P \left\{ \int_{E \cap B(x,r)} \left(\frac{\operatorname{dist}(y,P)}{r} \right)^p \frac{d\mathcal{H}^d(y)}{r^d} \right\}^{\frac{1}{p}}$$

  where the infimum is taken over all d-planes P.

Parametrizing

David-Toro 2012

Theorem (David - Toro, 2012)

Let $E \subseteq \mathbb{R}^n$ be a one-sided Reifenberg flat set. Then we can construct a map $\underline{f : \mathbb{R}^d \to \mathbb{R}^n}$, such that $E \subset f(\mathbb{R}^d)$ and f is bi-Hölder. Moreover, if we assume that there exists $M < \infty$ such that

$$\sum_{k \geq 0} \beta_\infty^E(x, r_k)^2 \leq M, \qquad \text{for all } x \in E,$$

then f is bi-Lipschitz.

Parametrizing
David-Toro 2012

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$$\sum_{k \geq 0} \beta_1^E(x, r_k)^2 \leq M, \qquad \text{for all } x \in E,$$

then f is bi-Lipschitz.

Parametrizing

The first main theorem I

Theorem (G., 2018)

Let $E \subseteq \mathbb{R}^n$ be a one-sided Reifenberg flat set and $\alpha \in (0,1)$. Also assume that there exists $M < \infty$ such that

$$\sum_{k \geq 0} \frac{\beta_{\infty}^{E}(x, r_k)^2}{r_k^{\alpha}} \leq M, \qquad \text{for all } x \in E. \tag{1}$$

Then we can construct a map $f : \mathbb{R}^d \to \mathbb{R}^n$, such that $E \subset f(\mathbb{R}^d)$ such that the map and its inverse are $C^{1,\alpha}$ continuous.
When $\alpha = 1$, if we replace r_k in the left hand side of (1) by $r_k \eta(r_k)$, where $\eta(r_k)^2$ satisfies the Dini condition, then we obtain that f and its inverse are $C^{1,1}$ maps.

Parametrizing

The first main theorem II

Theorem (G., 2018)

Let $E \subseteq \mathbb{R}^n$ be a one-sided Reifenberg flat set and $\alpha \in (0,1)$. Also assume that there exists $M < \infty$ such that

$$\sum_{k \geq 0} \frac{\beta_1^E(x, r_k)^2}{r_k^\alpha} \leq M, \qquad \text{for all } x \in E. \tag{2}$$

Then we can construct a map $f : \mathbb{R}^d \to \mathbb{R}^n$, such that $E \subset f(\mathbb{R}^d)$ such that the map and its inverse are $C^{1,\alpha}$ continuous.
When $\alpha = 1$, if we replace r_k in the left hand side of (2) by $r_k \eta(r_k)$, where $\eta(r_k)^2$ satisfies the Dini condition, then we obtain that f and its inverse are $C^{1,1}$ maps.

Why?

- Connection between smoothness and decay of β numbers (applications)

- Characterization of rectifiability of measures for different categories (TST type theorems) (Jones, Okikiolu, Schul, David-Semmes, Badger-Schul, Azzam-Tolsa+Tolsa, David-Schul, Li-Schul, Azzam-Schul, Edelen-Naber-Valtorta, Chousionis-Li-Zimmerman, Badger-Naples-Vellis, ...)

Rectifiability of measures

Theorem (G., 2018)

Let μ be a Radon measure on $\mathbb{R}^n$ such that $0 < \theta^{d*}(\mu, x) < \infty$ for μ-a.e. x and $\alpha \in (0,1)$. Assume that for μ-a.e. $x \in \mathbb{R}^n$,

$$J^{\mu}_{2,\alpha}(x) = \sum_{k \geq 0} \frac{\beta^{\mu}_{2}(x, r_k)^2}{r_k^{\alpha}} < \infty. \tag{3}$$

Then μ is (countably) $C^{1,\alpha}$ d-rectifiable.
When $\alpha = 1$, if we replace r_k in the left hand side of (3) by $r_k \eta(r_k)$, where $\eta(r_k)^2$ satisfies the Dini condition, then we obtain that E is C^2 rectifiable.

Rectifiability of measures

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(Works with Menger curvatures too! - Kolasiǹski, G.-Goering)

A $C^{1,\alpha}$ function

which is NOT $C^{1,\alpha+\varepsilon}$

Let h_J be the Haar wavelet, normalized so that $\int_J |h_J(x)|\, dx = 1$ and $\int_J h_J(x)\, dx = 0$, and define

$$\psi_I(x) = \int_{-\infty}^{x} h_I(t)\, dt$$

and

$$g_k(x) = \sum_{j=0}^{k} \sum_{J \in \Delta_j} 2^{-\alpha j} \psi_J(x),$$

where $\alpha \in (0,1)$. $g(x) = \lim_{k\to\infty} g_k(x)$ is a C^α function, and so $f(x) = \int_0^x g(t)\, dt$ is a $C^{1,\alpha}$ function.

A $C^{1,\alpha}$ function

which is NOT $C^{1,\alpha+\varepsilon}$

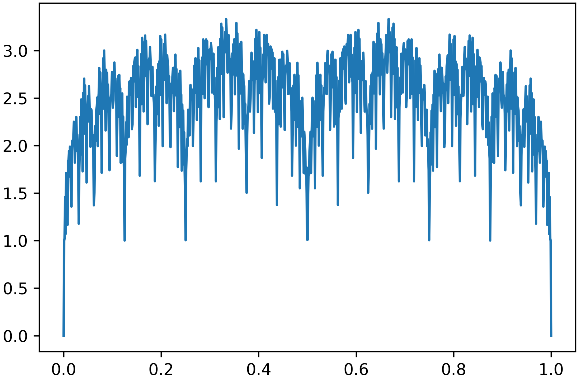

Figure: The function g_k on $[0, 1]$ for $k = 10$ and $\alpha = 0.0001$.

A $C^{1,\alpha}$ function

which is NOT $C^{1,\alpha+\varepsilon}$

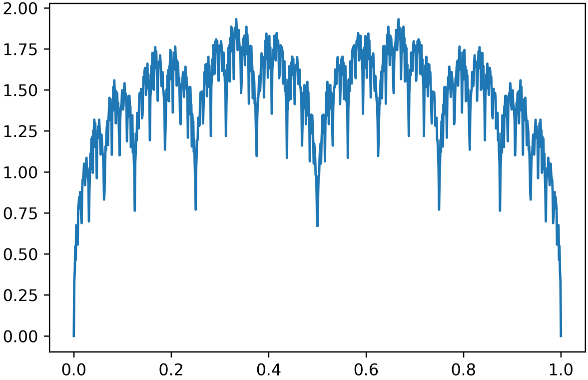

Figure: The function g_k on $[0,1]$ for $k = 10$ and $\alpha = 0.2$.

A $C^{1,\alpha}$ function

which is NOT $C^{1,\alpha+\varepsilon}$

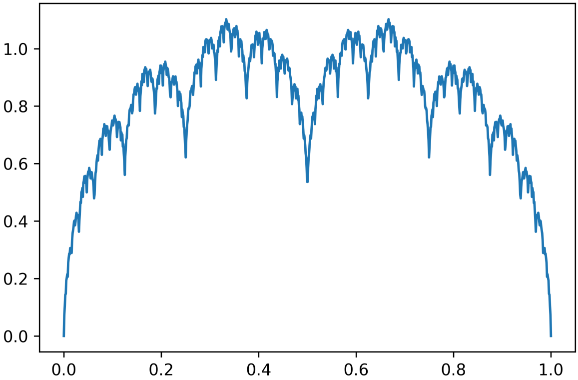

Figure: The function g_k on $[0, 1]$ for $k = 10$ and $\alpha = 0.5$.

A $C^{1,\alpha}$ function

which is NOT $C^{1,\alpha+\varepsilon}$

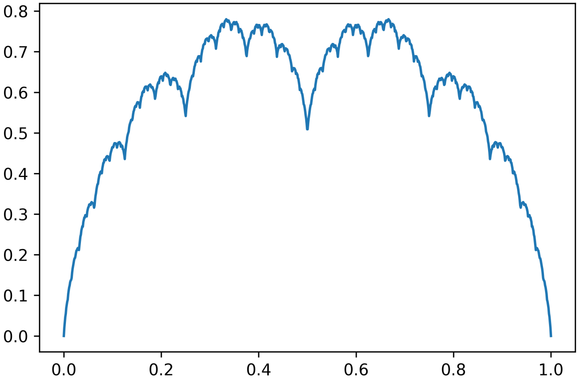

Figure: The function g_k on $[0,1]$ for $k = 10$ and $\alpha = 0.8$.

A $C^{1,\alpha}$ function

which is NOT $C^{1,\alpha+\varepsilon}$

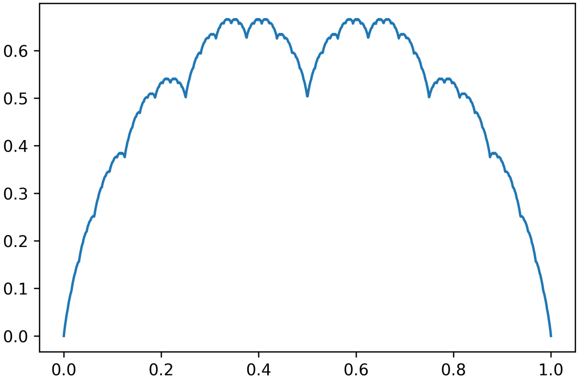

Figure: The function g_k on $[0,1]$ for $k = 10$ and $\alpha = 0.99$.

Thanks Alex and Sylvester!

Figure: Honu (green sea turtle) on Laniakea Beach, the other day.