Measurably Entire Functions and Their Growth

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The talk is partly based on a joint work with
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Translation Invariant Measures

Definitions

- Let $\mathcal{E}$ denote the space of entire functions, endowed with the topology of local uniform convergence.

- The group $\mathbb{C}$ acts on the space of entire functions by translations- for every $w \in \mathbb{C}$ and $F \in \mathcal{E}$ define:

$$(T_w F)(z) := F(z+w).$$

- Let λ be a probability measure on the Borel space associated with $\mathcal{E}$. We say λ is a **non-trivial translation invariant measure** if it is not supported on the constant functions and for every measurable set $A \subset \mathcal{E}$

$$\lambda(A) = \lambda\left(T_z^{-1} A\right).$$

Motivation & History

- We describe the growth of an entire function $f \in \mathcal{E}$ by

$$M_f(R) := \max_{z \in \overline{R\mathbb{D}}} |f(z)|,$$

where $R\mathbb{D} := \{z \in \mathbb{C},\ |z| < R\}$.

- Weiss showed that there are many non-trivial translation invariant probability measures on the set of entire functions, using ideas from dynamical systems.

- **Question:** [Weiss] What is the minimal possible growth of functions in the support of such measures?

Upper bound on the growth

Theorem (Buhovsky, G., Logunov, and Sodin
To appear in Journal d'Analyse Mathematique.)

(A) *There exists a non-trivial translation invariant probability measure λ on the space of entire functions such that for λ-almost every f, and for every $\varepsilon > 0$:*

$$\limsup_{R\to\infty} \frac{\log\log M_f(R)}{\log^{2+\varepsilon} R} = 0.$$

(B) *Let λ be a non-trivial translation invariant probability measure on the space of entire functions. Then for λ-almost every f, and for every $\varepsilon > 0$*

$$\lim_{R\to\infty} \frac{\log\log M_f(R)}{\log^{2-\varepsilon} R} = \infty.$$

Fat tails

Theorem (Buhovsky, G., Logunov, and Sodin
*To appear in Journal d'Analyse Mathematique.***)**

(A) There exists a non-trivial translation invariant probability measure λ *on the space of entire functions and a constant* $C > 0$ *such that for every* $t > 0$

$$\lambda\left(\left\{f \in \mathcal{E},\ \log_+ \log_+ |f(0)| > t\right\}\right) \leq \frac{C}{t}.$$

(B) For every non-trivial translation invariant probability measure λ *on the space of entire functions for every* $\varepsilon > 0$

$$\mathbb{E}\left(\log_+^{1+\varepsilon} \log_+ |f(0)|\right) = \infty.$$

Measurably Entire functions

Definitions

- Let $(X, \mathcal{B}, \mu)$ be a standard probability space.

- Denote by $PPT(X)$ the group of invertible probability preserving maps, $g : X \to X$ such that for every measurable set $A \in \mathcal{B}$

$$\mu(A) = \mu(g^{-1}(A)).$$

- A map $T : \mathbb{C} \to PPT(X)$ is called a **probability preserving action of** $\mathbb{C}$ (a $\mathbb{C}$-action in short) if it is a continuous homomorphism.

- A function $F : X \to \mathbb{C}$ is called **measurably entire** if it is a measurable non-constant function and for μ almost every $x \in X$ the function $f_x : \mathbb{C} \to \mathbb{C}$ defined by

$$f_x(z) := F(T_z x),$$

  is an entire function.

Connection to Translation Invariant Measures

- Every measurably entire function F on a standard probability space $(X, \mathcal{B}, \mu)$, induces a non-trivial translation invariant probability measure λ on the space of entire functions, by defining the measure

$$\lambda(A) := \mu\left(\{x \in X;\ f_x \in A\}\right),$$

where $f_x(z) := F(T_z x)$.

- For every translation invariant probability measure λ on $\mathcal{E}$, the space $(\mathcal{E}, \mathcal{B}, \lambda)$ is a standard probability space, and the function

$$F : \mathcal{E} \to \mathbb{C},\ F(g) = g(0)$$

is a measurably entire function.
! Note that $F(T_z g) = g(z)$.

Motivation

- **Question:** [Mackey] Does every $\mathbb{C}$-action on a standard probability space admits a measurably entire function?

- A $\mathbb{C}$-action, T, on a standard probability space $(X, \mathcal{B}, \mu)$ is called **free** if there exists $X_0 \subset X$, a measurable set of full measure, such that for every $x \in X_0$ and $z \in \mathbb{C}$

$$T_z x = x \Rightarrow z = 0.$$

  These are actions with no periodic points almost surely.

- **Theorem:** [Weiss, 1997] For every free probability preserving action of $\mathbb{C}$ on a standard probability space there exists a measurably entire function.

Growth

- **Question:** [Weiss] What is the minimal possible growth of measurably entire functions?

- There are two possible interpretations for this question:

(i) *What is the minimal growth of a measurably entire function of a $\mathbb{C}$-action on a standard probability space $(X, \mathcal{B}, \mu)$?*

(ii) *Given a $\mathbb{C}$-action on a standard probability space $(X, \mathcal{B}, \mu)$, what is the minimal growth of a measurably entire function?*

Growth cont.

- the first theorem stated in the previous section gives an ε-neighborhood full answer to the first interpretation:

- **Theorem:** [Buhovsky, G., Logunov, and Sodin,
To appear in Journal d'Analyse Mathematique.]

  (A) There exists a standard probability space $(\mathcal{E}, \mathcal{B}, \mu)$ *and a measurably entire function F such that for* μ *almost every* $x \in X$*, and for every* $\varepsilon > 0$*:*

$$\limsup_{R\to\infty} \frac{\log\log \max_{z\in\overline{R\mathbb{D}}} |F(T_z x)|}{\log^{2+\varepsilon} R} = 0.$$

  (B) For every standard probability space $(X, \mathcal{B}, \mu)$ *for every measurably entire function* $F: X \to \mathbb{C}$ μ*-almost every x, and for every* $\varepsilon > 0$

$$\lim_{R\to\infty} \frac{\log\log \max_{z\in\overline{R\mathbb{D}}} |F(T_z x)|}{\log^{2-\varepsilon} R} = \infty.$$

Results- the second interpretation

Theorem (G, *Isr. J. Math. (2019)*)

Let $(X,\mathcal{B},\mu)$ be a standard probability space, $T:\mathbb{C}\to PPT(X)$ be a free action. Then there exists a measurably entire function $F:X\to\mathbb{C}$ such that for every $\varepsilon>0$ for μ-almost every $x\in X$

$$\lim_{R\to\infty}\frac{\log\log\max_{z\in \overline{R\mathbb{D}}}|F(T_z x)|}{\log^{3+\varepsilon}R}=0.$$

- Note that there is a gap between the upper and lower bounds. Namely, it is not clear yet if there exists $p>2$ and a $\mathbb{C}$-action on a standard probability space such that for every measurably entire function $F:X\to\mathbb{C}$ for almost every x:

$$\lim_{R\to\infty}\frac{\log\log\max_{z\in \overline{R\mathbb{D}}}|F(T_z x)|}{\log^{p}R}=\infty.$$

Recurrently bounded functions

Idea of Proofs Leads to More Results

- Naive construction of a measure: Given a function f define the sequence of measures on $\mathcal{E}$

$$\mu_n(A) = \frac{1}{m(S_n)} \int_{S_n} \mathbf{1}_A \left(T_w f \right) dm(w),$$

where $A \subset \mathcal{E}$, $T_w f(z) := f(z+w)$, m denotes Haar's measure, and $S_n = [-a_n, a_n]^2$ for some sequence $\{a_n\} \nearrow \infty$.

- If a weakly converging subsequence exists, the limiting measure is translation invariant.
- For this measure not to be supported on $\{'\infty'\}$ and not to be supported on the constant functions f has to be "self similar".
- We need the same self-similarity of the function f for the general case as well.

Self Similar Entire functions or Bounded Subharmonic Functions

- Let u be a subharmonic function, and let

$$Z_u := \{z \in \mathbb{C}, u(z) \leq 0\} .$$

- We say the set Z_u is an $\varepsilon - R$ **recurrent set** if

$$\frac{m\left(B\left(z, R\left(|z|\right)\right) \cap Z_u\right)}{m\left(B\left(z, R\left(|z|\right)\right)\right)} \geq \varepsilon\left(|z|\right) .$$

  We say the function u is $\varepsilon - R$ **recurrently bounded function** if Z_u is an $\varepsilon - R$ recurrent set.

- We are interested in the cases where $R(\cdot)$ is a monotone increasing function with sub-linear growth and $\varepsilon(\cdot)$ is monotone decreasing function with decay slower than e^{-n}.
- **Question:** What can we say about the growth of such functions?

Recurrently Bounded Subharmonic Functions- Results

If $\varepsilon \in \left(0, \frac{1}{100}\right)$ is constant and $R(t) = 1$ is a constant function then we have to following result:

Theorem (Buhovsky, G., Logunov, and Sodin, *To appear in Journal d'Analyse Mathematique.***)**

(A) For every $\varepsilon \in \left(0, \frac{1}{100}\right)$ there exists a non-constant subharmonic $\varepsilon - 1$ recurrent function u such that

$$\limsup_{R\to\infty} \frac{\log M_u(R)}{R} < \infty.$$

(B) For every non-constant subharmonic $\varepsilon - 1$ recurrent function u,

$$\liminf_{R\to\infty} \frac{\log M_u(R)}{R} > 0.$$

Recurrently Bounded Subharmonic Functions- Results

Theorem (G. 2019)

Let $R : [0, \infty) \to \mathbb{R}_+$ be a C^2 monotone increasing concave function, such that $R(t) < c \cdot t + C$, for some $c \in (0, 1)$ and $C > 0$, and let $\delta \in \left(0, \frac{1}{100}\right)$.

(A) For every $\delta - R$ recurrent subharmonic function u

$$\liminf_{r \to \infty} \frac{\log M_u(r)}{\int_1^r \frac{1}{R(t)} dt} > 0.$$

(B) There exists a $\delta - R$ recurrent subharmonic function u such that

$$\limsup_{r \to \infty} \frac{\log M_u(r)}{\int_1^r \frac{1}{R(t)} dt} < \infty.$$

You're Welcome