Dimension and projections in normed spaces

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Orthogonal projections

$v \in S^1$ and $v^\perp$ a line in $\mathbb{R}^2$
$P_v : \mathbb{R}^2 \to v^\perp$ orthogonal projection onto $v^\perp$.

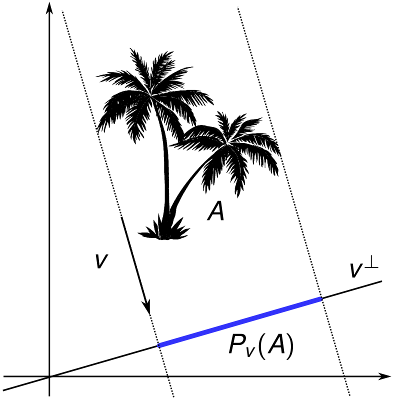
Given $A \subset \mathbb{R}^2$ Borel, $\dim A = s$
$\dim P_v(A) = ?$

Then trivially:

- $\dim P_v A \leq 1$
- $\dim P_v A \leq s$

Are these good bounds?

$\to$ Yes!



Orthogonal projections

In 1954, John Marstrand proved:

Given a Borel set $A \subset \mathbb{R}^2$, then for almost every line $v^\perp$

$$\dim P_v A = \min\{1, \dim A\}.$$

for almost every line: for all $v^\perp$ with $v \in E \subset S^1$, $\mathscr{H}^1(E) = 0$.

Marstrand's theorem rephrased:

$$\mathscr{H}^1\{\underbrace{v \in S^1 : \dim P_v A < \min\{1, \dim A\}}_{:=E}\} = 0$$

E is the set of exceptional directions for A.

Orthogonal projections

$A \subset \mathbb{R}^2$ a Borel set, $\dim A = s > 0$.

- Marstrand 1954:

  If $s \leq 1$, $\mathscr{H}^1\{v \in S^1 : \dim P_v A < s\} = 0$

  If $s > 1$, $\mathscr{H}^1\{v \in S^1 : \mathscr{H}^1(P_v(A)) = 0\} = 0$

- Kaufman 1968:

  If $s \leq 1$, $\dim\{v \in S^1 : \dim P_v A < s\} < s$.

- Mattila 1975: Generalization to projections of $\mathbb{R}^n$ to m-planes.

- Falconer 1982:

  If $s > 1$, $\dim\{v \in S^1 : \mathscr{H}^1(P_v(A)) = 0\} \leq 1 - s$

In codimension $= 1$ ...

Orthogonal projections

$A \subset \mathbb{R}^n$ a Borel set, $\dim A = s > 0$.

- Marstrand 1954:

  If $s \leq n-1$, $\mathscr{H}^{n-1}\{v \in S^{n-1} : \dim P_v A < s\} = 0$

  If $s > n-1$, $\mathscr{H}^{n-1}\{v \in S^{n-1} : \mathscr{H}^1(P_v(A)) = 0\} = 0$

- Kaufman 1968:

  If $s \leq n-1$, $\dim\{v \in S^1 : \dim P_v A < s\} < s$.

- Mattila 1975: Generalization to projections of $\mathbb{R}^n$ to m-planes.

- Falconer 1982:

  If $s > n-1$, $\dim\{v \in S^{n-1} : \mathscr{H}^{n-1}(P_v(A)) = 0\} \leq (n-1)-s$

In codimension $= 1$: Projections onto $(n-1)$-planes

Projections in normed spaces

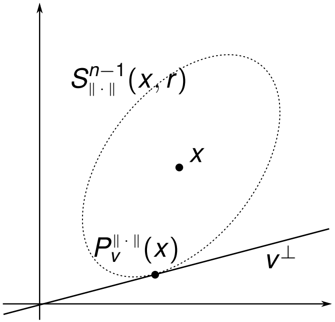
Let $\|\cdot\|$ be a strictly convex norm on $\mathbb{R}^n$.

Closest point projections onto $(n-1)$-planes are well defined:

Let $v \in S^{n-1}$ and $x \in \mathbb{R}^{n-1} \backslash v^{\perp}$
$r = \mathrm{dist}_{\|\cdot\|}(x, v^{\perp})$

$P_v^{\|\cdot\|} : \mathbb{R}^n \to v^{\perp}$
given by $P_v^{\|\cdot\|}(x) \in v^{\perp} \cap S^{n-1}_{\|\cdot\|}(x, r)$

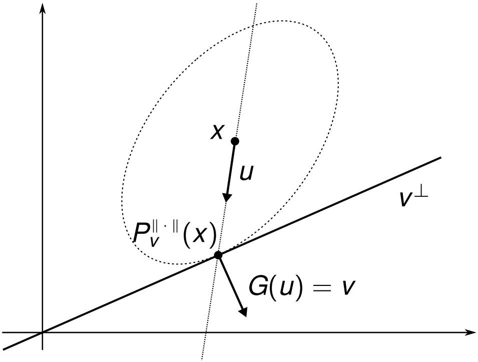
Do Marstrand-type projection theorems hold?



Projections in normed spaces

Let $\|\cdot\|$ be a strictly convex, C^1-regular norm on $\mathbb{R}^n$.

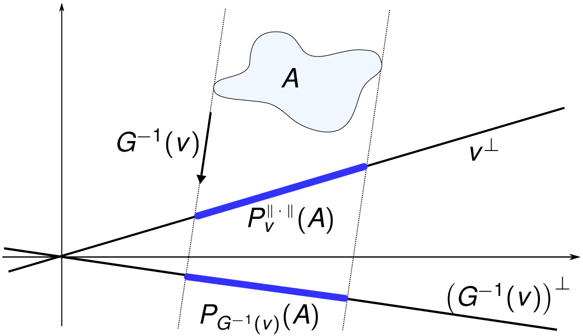
Gauss map $G : S^{n-1}_{\|\cdot\|} \to S^{n-1}$ is a homeomorphism.



For fixed $v \in S^{n-1}$, $P_v^{\|\cdot\|}$ is the projection onto $v^\perp$ in direction $G^{-1}(v)$.

Projection Theorems for normed spaces

How to prove Marstrand-type results for projections $P_v^{\|\cdot\|} : \mathbb{R}^n \to v^{\perp}$?



$P_v^{\|\cdot\|}(A) \sim P_{G^{-1}(v)}(A)$ in measure and dimension $\Rightarrow$ $E_{\|\cdot\|} = G(E)$

Projection Theorems for normed spaces

Theorem 1. $\|\cdot\|$ a strictly convex C^1-regular norm on $\mathbb{R}^n$ and G is dimension non-increasing and preserves $\mathscr{H}^{n-1}$-zero sets. Then for every Borel set $A \subseteq \mathbb{R}^n$ with $\dim A = s$,

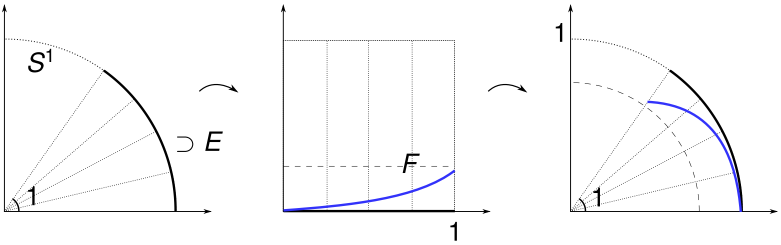
- if $s \leq n-1$
  - $\mathscr{H}^{n-1}\{v \in S^{n-1} : \dim P_v^{\|\cdot\|} A < s\} = 0$
  - $\dim\{v \in S^1 : \dim P_v^{\|\cdot\|} A < t\} < t,\ 0 < t \leq s.$
- if $s > n-1$,
  - $\mathscr{H}^{n-1}\{v \in S^{n-1} : \mathscr{H}^1(P_v^{\|\cdot\|}(A)) = 0\} = 0$
  - $\dim\{v \in S^{n-1} : \mathscr{H}^{n-1}(P_v^{\|\cdot\|}(A)) = 0\} \leq (n-1) - s$

Note: If $\|\cdot\|$ is $C^{1,1}$-regular and strictly convex, then Theorem 1 holds.

Projection Theorems for normed spaces

Theorem 2. There exists a strictly convex and C^1-regular norm $\|\cdot\|$ on $\mathbb{R}^2$ such that Theorem 1 fails.

Proof: Find $\|\cdot\|$ strictly convex and C^1-regular and $A \subset \mathbb{R}^2$ Borel such that G blows up E in measure and dimension.



Open problems

Summary:

- If $\|\cdot\|$ is $C^{1,1}$ $\Rightarrow$ projection theorems hold in $\mathbb{R}^n$
- If $\|\cdot\|$ only C^1 $\Rightarrow$ counterexample in $\mathbb{R}^2$

Question: What about $C^{1,\delta}$, $0 < \delta < 1$?

Answer: Depends on the structure of the exceptional sets E, which is in general unknown.

Question: Counterexample for $n > 2$? $\rightarrow$ work in progress.

Thank you!