Modulus of sets of finite perimeter and quasiconformal maps between metric spaces of globally Q-bounded geometry

Rebekah Jones*, Panu Lahti, Nageswari Shanmugalingam

March 22, 2019

Main Topic

- It is well-known in $\mathbb{R}^n$ that a homeomorphism $f : \Omega \to \Omega'$ is quasiconformal if and only if there exists $K \geq 1$ such that

$$\frac{1}{K}\mathrm{Mod}_n(\Gamma) \leq \mathrm{Mod}_n(f\Gamma) \leq K\mathrm{Mod}_n(\Gamma)$$

  holds for all curve families Γ in Ω.

Main Topic

- It is well-known in $\mathbb{R}^n$ that a homeomorphism $f : \Omega \to \Omega'$ is quasiconformal if and only if there exists $K \geq 1$ such that

$$\frac{1}{K}\mathrm{Mod}_n(\Gamma) \leq \mathrm{Mod}_n(f\Gamma) \leq K\mathrm{Mod}_n(\Gamma)$$

  holds for all curve families Γ in Ω.

- Also, a quasiconformal map quasipreserves the $\frac{n}{n-1}$-modulus of surfaces. (Kelly, 1973)

Main Topic

- It is well-known in $\mathbb{R}^n$ that a homeomorphism $f : \Omega \to \Omega'$ is quasiconformal if and only if there exists $K \geq 1$ such that

$$\frac{1}{K}\mathrm{Mod}_n(\Gamma) \leq \mathrm{Mod}_n(f\Gamma) \leq K\mathrm{Mod}_n(\Gamma)$$

  holds for all curve families Γ in Ω.

- Also, a quasiconformal map quasipreserves the $\frac{n}{n-1}$-modulus of surfaces. (Kelly, 1973)

- Does an analogous result hold in the setting of metric measure spaces?

Main Topic

- It is well-known in $\mathbb{R}^n$ that a homeomorphism $f : \Omega \to \Omega'$ is quasiconformal if and only if there exists $K \geq 1$ such that

$$\frac{1}{K}\mathrm{Mod}_n(\Gamma) \leq \mathrm{Mod}_n(f\Gamma) \leq K\mathrm{Mod}_n(\Gamma)$$

  holds for all curve families Γ in Ω.

- Also, a quasiconformal map quasipreserves the $\frac{n}{n-1}$-modulus of surfaces. (Kelly, 1973)

- Does an analogous result hold in the setting of metric measure spaces?

  Answer: Yes, with some standard geometric restrictions on the spaces.

The Setting

- (X, d_X, μ_X) is a complete, proper metric measure space

The Setting

- (X, d_X, μ_X) is a complete, proper metric measure space
- μ_X is an Ahlfors Q-regular measure ($Q > 1$):

  There exists a constant $C_A \geq 1$ such that for each $x \in X$ and $0 < r < 2\text{diam}(X)$,

$$\frac{r^Q}{C_A} \leq \mu_X(B(x,r)) \leq C_A\, r^Q.$$

The Setting

- (X, d_X, μ_X) is a complete, proper metric measure space
- μ_X is an Ahlfors Q-regular measure ($Q > 1$):

  There exists a constant $C_A \geq 1$ such that for each $x \in X$ and $0 < r < 2\mathrm{diam}(X)$,

$$\frac{r^Q}{C_A} \leq \mu_X(B(x,r)) \leq C_A\, r^Q.$$

- Let $Q^* = \frac{Q}{Q-1}$.

Curves and Upper Gradients

A curve is a continuous function $\gamma : [a, b] \to X$, parametrized by arc length.

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces. A non-negative Borel function g on X is an *upper gradient* of $f : X \to Y$ if for all curves γ,

$$d_Y\big(f(\gamma(a)), f(\gamma(b))\big) \leq \int_\gamma g\, ds,$$

Curves and Upper Gradients

A curve is a continuous function $\gamma : [a, b] \to X$, parametrized by arc length.

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces. A non-negative Borel function g on X is an *upper gradient* of $f : X \to Y$ if for all curves γ,

$$d_Y\big(f(\gamma(a)), f(\gamma(b))\big) \leq \int_\gamma g \, ds,$$

Ex: For a $\mathcal{C}^1$ function $f : \mathbb{R}^n \to \mathbb{R}$, $|\nabla f|$ is an upper gradient of f by the FTC.

Poincaré Inequality

Definition

The space X supports a 1-Poincaré inequality if there exist constants $C > 0$ and $\lambda \geq 1$ such that for all functions $u \in L^1_{loc}(X)$, all upper gradients g of u and all balls $B \subset X$, we have

$$\fint_B |u - u_B| \, d\mu \leq C \, rad(B) \left(\fint_{\lambda B} g \, d\mu \right).$$

Here

$$u_B := \fint_B u \, d\mu := \frac{1}{\mu(B)} \int_B u \, d\mu.$$

Modulus of curves

Definition

Let Γ be a collection of curves on X. The admissible class of Γ, denoted $\mathcal{A}(\Gamma)$, is the set of all Borel measurable functions $\rho : X \to [0, \infty]$ such that

$$\int_\gamma \rho \; ds \geq 1$$

for all $\gamma \in \Gamma$. Then

$$\mathrm{Mod}_p(\Gamma) := \inf_{\rho \in \mathcal{A}(\Gamma)} \int_X \rho^p \; d\mu_X.$$

Modulus of Measures

Definition

Let $\mathcal{L}$ be a collection of measures on X. The admissible class of $\mathcal{L}$, denoted $\mathcal{A}(\mathcal{L})$, is the set of all Borel measurable functions $\rho : X \to [0, \infty]$ such that

$$\int_X \rho \, d\lambda \geq 1$$

for all $\lambda \in \mathcal{L}$. Then

$$\mathrm{Mod}_p(\mathcal{L}) := \inf_{\rho \in \mathcal{A}(\mathcal{L})} \int_X \rho^p \, d\mu_X .$$

Modulus of Measures

Definition

Let $\mathcal{L}$ be a collection of measures on X. The admissible class of $\mathcal{L}$, denoted $\mathcal{A}(\mathcal{L})$, is the set of all Borel measurable functions $\rho : X \to [0, \infty]$ such that

$$\int_X \rho \, d\lambda \geq 1$$

for all $\lambda \in \mathcal{L}$. Then

$$\mathrm{Mod}_p(\mathcal{L}) := \inf_{\rho \in \mathcal{A}(\mathcal{L})} \int_X \rho^p \, d\mu_X.$$

- If we take $\mathcal{L} = \{ds\lfloor_\gamma : \gamma \in \Gamma\}$, we get back p-modulus of curves.

Measure Theoretic Boundary

For a measurable set $E \subset X$ and $x \in X$, we define the *upper* and *lower measure densities* (respectively) of E at x by

$$\overline{D}(E, x) = \limsup_{r \to 0^+} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))}$$

$$\underline{D}(E, x) = \liminf_{r \to 0^+} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))}.$$

Measure Theoretic Boundary

For a measurable set $E \subset X$ and $x \in X$, we define the *upper* and *lower measure densities* (respectively) of E at x by

$$\overline{D}(E,x) = \limsup_{r \to 0^+} \frac{\mu(B(x,r) \cap E)}{\mu(B(x,r))}$$

$$\underline{D}(E,x) = \liminf_{r \to 0^+} \frac{\mu(B(x,r) \cap E)}{\mu(B(x,r))}.$$

The *measure theoretic boundary* of E is

$$\partial^* E = \{x \in X : \overline{D}(E,x) > 0 \text{ and } \overline{D}(X \setminus E, x) > 0\}.$$

Sets of Finite Perimeter

Definition (Perimeter measure)

Let $E \subset X$ Borel and $U \subset X$ open. Then

$$P(E,U) = \inf\left\{ \liminf_{n\to\infty} \int_U g_{u_n} d\mu : \mathrm{Lip}_{loc}(U) \ni u_n \to \chi_E \text{ in } L^1_{loc}(U) \right\}.$$

- We say that E is of *finite perimeter* if $P(E,X) < \infty$.
- If E is of finite perimeter, then $P(E,\cdot)$ defines a Radon measure on X. (Miranda, 2003)
- $P(E,\cdot)$ is supported on a subset of $\partial^* E$ when X supports a Poincaré inequality.

Σ-boundary

Let

$$\Sigma E := \{x \in \partial^* E : \underline{D}(E, x) > 0 \text{ and } \underline{D}(X \setminus E, x) > 0\}\,.$$

Theorem (Ambrosio, 2002)

For any set $E \subset X$ of finite perimeter:

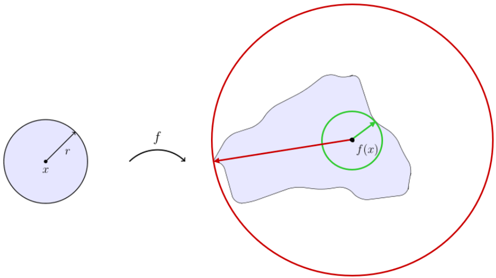
- *the perimeter measure $P(E, \cdot)$ is concentrated on ΣE*
- $\mathcal{H}^{Q-1}(\partial^* E \setminus \Sigma E) = 0$
- $P(E, \cdot) \simeq \mathcal{H}^{Q-1} \llcorner_{\Sigma E}$

L_f and ℓ_f

For a homeomorphism $f : X \to Y$, we define

$$L_f(x,r) := \sup_{y \in \overline{B}(x,r)} d_Y(f(x), f(y)) \text{ and } L_f(x) := \limsup_{r \to 0} \frac{L_f(x,r)}{r}$$

$$\ell_f(x,r) := \inf_{y \in X \setminus B(x,r)} d_Y(f(x), f(y)) \text{ and } \ell_f(x) := \liminf_{r \to 0} \frac{\ell_f(x,r)}{r}.$$

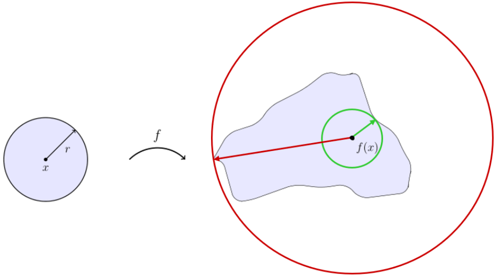


Quasiconformal Map

Definition

The function $f : X \to Y$ is *quasiconformal (QC)* if there is a constant $K \geq 1$ such that for all $x \in X$ we have

$$\limsup_{r \to 0^+} \frac{L_f(x, r)}{\ell_f(x, r)} \leq K.$$



A Result in $\mathbb{R}^n$

Theorem (Kelly, 1973)

Let $\Omega, \Omega' \subset \mathbb{R}^n$ and let $\mathcal{P}$ be a collection of surfaces in Ω. If $f : \Omega \to \Omega'$ is quasiconformal then

$$\frac{1}{C}\mathrm{Mod}_{\frac{n}{n-1}}(\mathcal{P}) \leq \mathrm{Mod}_{\frac{n}{n-1}}(f\mathcal{P}) \leq C\mathrm{Mod}_{\frac{n}{n-1}}(\mathcal{P}).$$

- Here a surface is the boundary of a Lebesgue measurable set $E \subset \Omega$ with $\mathcal{H}^{n-1}(\partial E) < \infty$ which also satisfies a certain double-sided cone condition.
- With this definition, $\mathrm{Mod}_{n/(n-1)}$-almost every surface in Ω gets mapped to a surface in Ω' under f.

Assumptions

- X and Y are complete, Ahlfors Q-regular and support a 1-Poincaré inequality.
- $f : X \to Y$ is a QC map.
- For a collection of sets of finite perimeter $\mathcal{F}$, we consider the measures
$$\mathcal{L} = \left\{ \mathcal{H}^{Q-1} \lfloor_{\Sigma E} : E \in \mathcal{F} \right\}$$
and
$$f\mathcal{L} = \left\{ \mathcal{H}^{Q-1} \lfloor_{\Sigma f(E)} : E \in \mathcal{L} \right\}.$$

Main Result

Theorem (J., Lahti, Shanmugalingam)

There exists $C > 0$ such that for every collection of bounded sets of postive and finite perimeter in X, we have that

$$\mathrm{Mod}_{Q^*}(\mathcal{L}) \leq C\,\mathrm{Mod}_{Q^*}(f\mathcal{L})$$

and

$$\mathrm{Mod}_{Q^*}(f\mathcal{L}') \leq C\,\mathrm{Mod}_{Q^*}(\mathcal{L}')$$

where $\mathcal{L}'$ consists of all $E \in \mathcal{L}$ for which $0 < L_f(x) < \infty$ for $\mathcal{H}^{Q-1}$-almost every $x \in \Sigma E$.

Main Ingredients of the Proof

- Uniform density property (Korte Marola Shanmugalingam, 2012): A QC map f preserves the measure density of points, so $f(\Sigma E) = \Sigma f(E)$.

Main Ingredients of the Proof

- Uniform density property (Korte Marola Shanmugalingam, 2012): A QC map f preserves the measure density of points, so $f(\Sigma E) = \Sigma f(E)$.

- Absolute continuity:

$$f_{\#}\mathcal{H}^{Q-1}\lfloor_{\Sigma f(E)} \ll \mathcal{H}^{Q-1}\lfloor_{\Sigma E}$$

  which gives a $(Q-1)$-change of variables formula via the Radon-Nikodym Theorem.

Main Ingredients of the Proof

- Uniform density property (Korte Marola Shanmugalingam, 2012): A QC map f preserves the measure density of points, so $f(\Sigma E) = \Sigma f(E)$.

- Absolute continuity:

$$f_{\#}\mathcal{H}^{Q-1}\lfloor_{\Sigma f(E)} \ll \mathcal{H}^{Q-1}\lfloor_{\Sigma E}$$

  which gives a $(Q-1)$-change of variables formula via the Radon-Nikodym Theorem.

- The comparison:

$$J_{f,E}(x) \leq C\, J_f(x)^{(Q-1)/Q}.$$

Main Ingredients of the Proof

- Uniform density property (Korte Marola Shanmugalingam, 2012): A QC map f preserves the measure density of points, so $f(\Sigma E) = \Sigma f(E)$.

- Absolute continuity:

$$f_{\#}\mathcal{H}^{Q-1}\lfloor_{\Sigma f(E)} \ll \mathcal{H}^{Q-1}\lfloor_{\Sigma E}$$

  which gives a $(Q-1)$-change of variables formula via the Radon-Nikodym Theorem.

- The comparison:

$$J_{f,E}(x) \leq C\, J_f(x)^{(Q-1)/Q}.$$

Using these we take a function admissible for computing $\mathrm{Mod}_{Q^*}(f\mathcal{L})$ and "pull it back" to X, get an admissible function for computing $\mathrm{Mod}_{Q^*}(\mathcal{L})$ and use it to estimate the modulus.

Converse

Theorem

Suppose $f : X \to Y$ is a homeomorphism and there exists $C \geq 1$ such that for any collection $\mathcal{L}$ of sets E in X for which $f(E)$ is of finite perimeter in Y,

$$\mathrm{Mod}_{\frac{Q}{Q-1}}(f\mathcal{L}) \leq C\mathrm{Mod}_{\frac{Q}{Q-1}}(\mathcal{L}). \tag{1}$$

Then f is quasiconformal.

Converse

To prove the converse we use the following proposition on the sets $E := f^{-1}(B(x, \ell_f(x,r)))$ and $F := f^{-1}(B(x, L_f(x,r)))$.

Proposition

There exists $C > 0$ such that for any open $\Omega \subset X$ and any non-empty, closed and disjoint $E, F \subset X$,

$$\frac{1}{C} \leq \left(\mathrm{Mod}_{\frac{Q}{Q-1}}(\mathcal{L})\right)^{\frac{Q-1}{Q}} \left(\mathrm{Mod}_Q(\Gamma)\right)^{\frac{1}{Q}} \leq C$$

where Γ is the collection of curves that start in E and end in F and $\mathcal{L}$ is the collection of measurable sets that separate the capacitary thickness points of E and F.