The quasiconformal geometry of continuum trees

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Quasiconformal geometry I

The **quasiconformal (=qc) geometry** of a metric space is related to geometric conditions that are robust under changes of scale, or depend only on *relative* distances (=ratios of distances).

- **Example 1:** A metric space X is *doubling* if there ex. $N \in \mathbb{N}$ such that every ball in X can be covered by N (or fewer) balls of half the radius.
- **Example 2:** A metric space (X, d) is *of bounded turning* if there ex. $\lambda \geq 1$ such any two points $x, y \in X$ can be joined by a continuum $E \subseteq X$ s.t.

$$\operatorname{diam} E \leq \lambda d(x, y).$$

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Quasconformal geometry II

- Relevant maps in qc-geometry:
  quasisymmetric (=qs) homeomorphism $f\colon X \to Y$.
  Qs-homeos. distort relative distances in a controlled way; they map metric balls to "quasi-balls" (=sets with uniformly bounded "eccentricity").

- Conditions in qc-geometry are typically invariant under quasisymmetries.

  Example: Let $f\colon X \to Y$ be a quasisymmetry.
  If X is doubling, then Y is doubling.
  If X is of bdd. turning, then Y is of bdd. turning.

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Quasconformal geometry III

- Qc-geometry is important for the study of *self-similar* fractals.
- Strong motivation from complex dynamics or geometric group theory (Thurston's s characterization of postcritically-finite rational maps; Cannon's conjecture).
- Program to systematically study qc-geometry of low-dimensional fractals such as:
  Cantor sets, quasi-circles, Sierpiński gaskets and carpets, fractal 2-spheres, dendrites or continuum trees, etc.

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Continuum trees or dendrites

- A *(continuum) tree* or *dendrite* is a locally connected, connected, compact metric space s.t. any two points can be joined by a unique arc.

Trees appear in various contexts:

- as Julia sets,
- as attractors of iterated function systems (e.g., the CSST=continuum self-similar tree),
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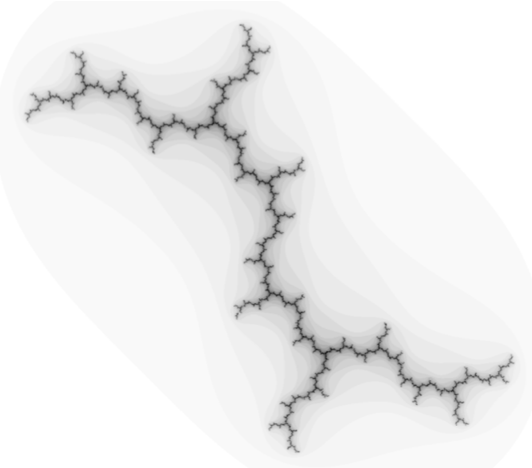
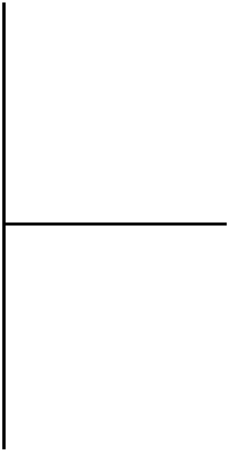
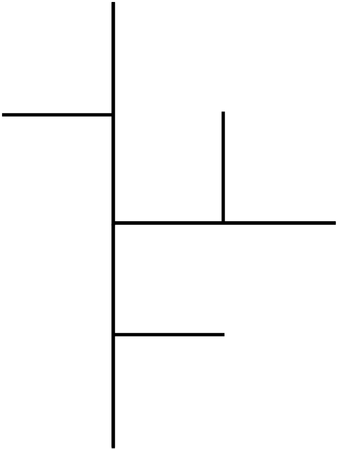
A Julia set



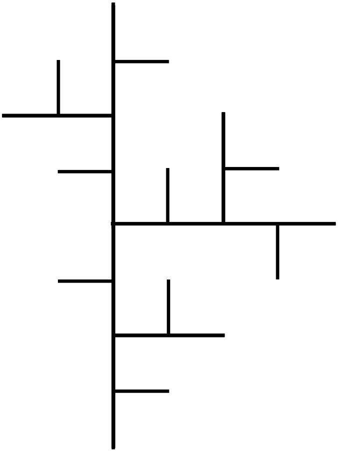
The Julia set $\mathcal{J}(P)$ of $P(z) = z^2 + i$ (= set of points with bounded orbit under iteration) is a tree.

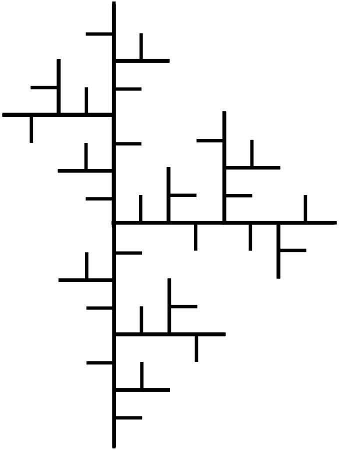
CSST $\mathbb{T}$ (=continuum self-similar tree)

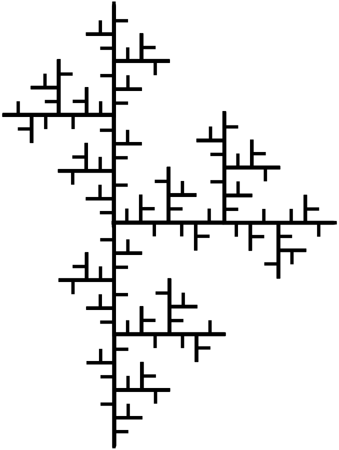




CSST $\mathbb{T}$







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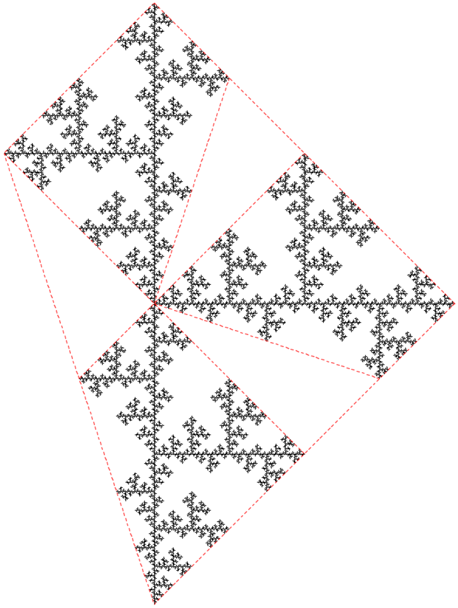

The CSST $\mathbb{T}$ is:

- a geodesic continuum tree (as an abstract metric space).
- an attractor of an iterated function system (as a subset of the plane). Define

$$f_1(z) = \frac{1}{2}z - \frac{i}{2}, \quad f_2(z) = -\frac{1}{2}\bar{z} + \frac{i}{2}, \quad f_3(z) = \frac{i}{2}\bar{z} + \frac{1}{2},$$

Then $\mathbb{T} \subseteq \mathbb{C}$ is the unique non-empty compact set satisfying

$$\mathbb{T} = f_1(\mathbb{T}) \cup f_2(\mathbb{T}) \cup f_3(\mathbb{T}).$$

So $\mathbb{T}$ is the attractor of the iterated function system $\{f_1, f_2, f_3\}$ in the plane.

M. Bonk and Huy Tran, *The continuum self-similar tree*, on arXiv.

Theorem (Charatonik-Dilks 1994; B.-Tran 2018)

A continuum tree T is homeomorphic to the CCST $\mathbb{T}$ iff all branch points of T have order 3 and they are dense in T.

Theorem (Croyden-Hambly 2008; B.-Tran 2018)

The continuum random tree *(CRT) is almost surely homeomorphic to the CSST.*

Theorem (B.-Tran)

The Julia set $\mathcal{J}(z^2 + i)$ is homeomorphic to the CSST.

Topology of the CSST

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Problems about the quasiconformal geometry of trees

- When can one promote a homeomorphism between trees to a quasisymmetry?
- Is there a characterization of the CSST up to qs-equivalence?
- What can one say about the qc-geometry of the CRT or Julia sets of postcritically-finite polynomials?
- Are there some canonical models for certain classes of trees up to qs-equivalence (uniformization problem)?

Qs-characterization of quasi-arcs

Theorem (Tukia-Väisälä 1980)

Let α be a metric arc. Then α is qs-equivalent to $[0,1]$ iff α is doubling and of bounded turning.

A metric space X is *doubling* is there exists $N \in \mathbb{N}$ such that every ball in X can be covered by N (or fewer) balls of half the radius.

A metric space (X, d) is *of bounded turning* if there ex. $\lambda \geq 1$ such any two points $x, y \in X$ can be joined by a continuum $E \subseteq X$ s.t.

$$\operatorname{diam} E \leq \lambda d(x, y).$$

Qs-uniformization of quasi-trees

M. Bonk and D. Meyer, *Quasi-trees and geodesic trees*, on arXiv.

Theorem (B.-Meyer 2019)

Let T be a tree that is doubling and of bounded turning (a "quasi-tree"). Then T is qs-equivalent to a geodesic tree.

A tree (T, d) is *geodesic* if for all $x, y \in T$,

$$\text{length}\,[x, y] = d(x, y).$$

Note: in a quasi-tree we have $\text{diam}\,[x, y] \asymp d(x, y)$.

Proof of Theorem: For each level n decompose the metric space (T, d) into pieces X^n with $\text{diam}\, X^n \asymp \delta^n$, where $\delta > 0$ is a small parameter. Carefully redefine metric d by assigning new diameters to these pieces to obtain a geodesic metric ϱ on T. With suitable choices, the identity map $(T, d) \to (T, \varrho)$ is a quasisymmetry. □

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Height of a branch point

Let T be a tree. Then $p \in T$ is a *branch point* of T if $T \setminus \{p\}$ has at least three connected components $B_1, B_2, B_3, \ldots$ (the *branches* of T at p).
One can order them so that

$$\operatorname{diam} B_1 \geq \operatorname{diam} B_2 \geq \operatorname{diam} B_3 \geq \ldots .$$

Then the *height* of p in T is defined as

$$\begin{aligned} h_T(p) &= \operatorname{diam} B_3 \\ &= \text{diameter of third largest component of } T \backslash \{p\}. \end{aligned}$$

Characterization of the CSST up to qs-equivalence

Theorem (B.-Meyer)

Let (T, d) be tree. Then T is qs-equivalent to the CSST if and only if the following conditions are true:

- *T is a quasi-tree, i.e., doubling and of bounded turning.*
- *each branch point of T has order 3.*
- *the branch points of T are* uniformly relatively separated, *i.e., if $p, q \in T$ are branch points, then*

$$d(p, q) \gtrsim \min\{h_T(p), h_T(q)\}.$$

- *the branch points of T are* uniformly dense, *i.e., if $a, b \in T$ are distinct points, then there exists a branch point $p \in [a, b]$ s.t.*

$$h_T(p) \gtrsim d(a, b).$$

Some ideas for the proof of Theorem:

Necessity (easy): the CSST has the stated properties and they are invariant under quasisymmetries.

Sufficiency: Carefully decompose the given tree (T, d) into pieces. Obtain a sequence of finite trivalent trees T_n with good geometric control that better and better approximate T, i.e., $T_n \to T$. Use universality property of the CSST to find trees $T'_n \subseteq \mathbb{T}$ s.t. T'_n is uniformly qs-equivalent to T_n. Pass to limit $n \to \infty$ to find a quasisymmetry $T \to \mathbb{T}$.

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Question

Is the CRT almost surely qs-equivalent to the CSST?

No, because the CSST is a quasi-tree, and in particular doubling, while the CRT is not doubling, and doubling is preserved under quasisymmetries.

Open Problem (very hard!)

Are two independent samples of the CRT almost surely qs-equivalent?

General theme: Geometric uniqueness of probabilistic models. Known (up to quasi-isometric equivalence) for Bernoulli percolation on $\mathbb{Z}$, Poisson point process on $\mathbb{R}$ (Basu-Sly).

Quasiconformal geometry of the CRT

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