# Singular sets of UAD measures

Abdalla Dali Nimer (University of Chicago)

AMS Spring Central and Western Sectional Meeting, Hawaii

March 23rd, 2019

**Besicovitch(1938)** Let $E \subset \mathbb{R}^2, 0 < \mathcal{H}^1(E) < \infty$ and for $\mathcal{H}^1$ almost every $x \in E$,

$$\lim_{r \to 0} \frac{\mathcal{H}^1(E \cap B(x,r))}{2r} = 1.$$

Then

$$E \text{ is } 1 - \text{rectifiable.}$$

## Theorem (Preiss)

*Let $\Phi$ be a Radon measure on $\mathbb{R}^d$. Then $\Phi$ is n-rectifiable (i.e. $\Phi << \mathcal{H}^n$ and that $\Phi(\mathbb{R}^d \backslash E) = 0$ for some n-rectifiable set E) if and only if for $\Phi$ almost every $x$, $\Theta^n(\Phi, x) = \lim_{r \to 0} \frac{\Phi(B(x,r))}{\omega_n r^n}$ exists and*

$$0 < \Theta^n(\Phi, x) < \infty.$$

## Definition

Let $\Phi$ be a Radon measure on $\mathbb{R}^d$, $x$ a point in its support. We say that $\lambda$ is a pseudo-tangent measure of $\Phi$ at $x$ if $\lambda \neq 0$ and there exists sequences of positive reals $(r_i)$,$(c_i)$ with $r_i \downarrow 0$ and a sequence of points $x_i$, $x_i \to x$ such that:

$$c_i T_{x_i,r_i}[\Phi] \rightharpoonup \lambda \text{ as } i \to \infty,$$

where the convergence is the weak convergence of measures and $c_i T_{x,r}[\Phi]$ is the push-forward of $\Phi$ by the homothecy $T_{x,r}(y) = \frac{y-x}{r}$.

## Definition

Let $\mu$ be a Radon measure in $\mathbb{R}^d$.

- We say $\mu$ is $n$-uniform if there exists $c > 0$ such that for all $x \in spt(\mu)$, $r > 0$:

$$\mu(B(x,r)) = cr^n.$$

- We say $\mu$ is uniformly distributed or uniform if there exists a function $f : (0,+\infty) \to (0,+\infty)$ such that: for all $x \in spt(\mu)$, $r > 0$:

$$\mu(B(x,r)) = f(r).$$

## Definition

Let $\mu$ be a Radon measure in $\mathbb{R}^d$.

- We say $\mu$ is $n$-uniform if there exists $c > 0$ such that for all $x \in spt(\mu)$, $r > 0$:

$$\mu(B(x, r)) = cr^n.$$

- We say $\mu$ is uniformly distributed or uniform if there exists a function $f : (0, +\infty) \to (0, +\infty)$ such that: for all $x \in spt(\mu)$, $r > 0$:

$$\mu(B(x, r)) = f(r).$$

- (**Preiss**)The support of an 1-uniform measure is a line, of a 2-uniform measure is a plane.
- (**Kirchheim-Preiss**)The support of an uniform measure is an analytic variety.
- (**Kowalski-Preiss**) The support of an $n$-uniform measure in $\mathbb{R}^{n+1}$ can only be an $n$-plane or (up to rotation) $\mathbb{R}^{n-3} \times C$ where $C = \{(x_1, x_2, x_3, x_4); x_4^2 = x_1^2 + x_2^2 + x_3^2\}$ .
- (**N.**) $\mu$ is an $n$-uniform measure in $\mathbb{R}^d$, $n \geq 3$ and $\mathcal{S}_\mu$ its set of singularities. Then $dim(\mathcal{S}_\mu) \leq n - 3$.

- (**Preiss**)The support of an 1-uniform measure is a line, of a 2-uniform measure is a plane.
- (**Kirchheim-Preiss**)The support of an uniform measure is an analytic variety.
- (**Kowalski-Preiss**) The support of an $n$-uniform measure in $\mathbb{R}^{n+1}$ can only be an $n$-plane or (up to rotation) $\mathbb{R}^{n-3} \times C$ where $C = \{(x_1, x_2, x_3, x_4); x_4^2 = x_1^2 + x_2^2 + x_3^2\}$ .
- (**N.**) $\mu$ is an $n$-uniform measure in $\mathbb{R}^d$, $n \geq 3$ and $\mathcal{S}_\mu$ its set of singularities. Then $dim(\mathcal{S}_\mu) \leq n - 3$.

- (**Preiss**)The support of an 1-uniform measure is a line, of a 2-uniform measure is a plane.
- (**Kirchheim-Preiss**)The support of an uniform measure is an analytic variety.
- (**Kowalski-Preiss**) The support of an $n$-uniform measure in $\mathbb{R}^{n+1}$ can only be an $n$-plane or (up to rotation) $\mathbb{R}^{n-3} \times C$ where $C = \left\{(x_1, x_2, x_3, x_4); x_4^2 = x_1^2 + x_2^2 + x_3^2\right\}$ .
- (**N.**) $\mu$ is an $n$-uniform measure in $\mathbb{R}^d$, $n \geq 3$ and $\mathcal{S}_\mu$ its set of singularities. Then $dim(\mathcal{S}_\mu) \leq n - 3$.

- (**Preiss**)The support of an 1-uniform measure is a line, of a 2-uniform measure is a plane.
- (**Kirchheim-Preiss**)The support of an uniform measure is an analytic variety.
- (**Kowalski-Preiss**) The support of an $n$-uniform measure in $\mathbb{R}^{n+1}$ can only be an $n$-plane or (up to rotation) $\mathbb{R}^{n-3} \times C$ where $C = \{(x_1, x_2, x_3, x_4); x_4^2 = x_1^2 + x_2^2 + x_3^2\}$ .
- (**N.**) $\mu$ is an $n$-uniform measure in $\mathbb{R}^d$, $n \geq 3$ and $\mathcal{S}_\mu$ its set of singularities. Then $dim(\mathcal{S}_\mu) \leq n - 3$.

## Definition

Consider a Radon measure $\mu$ on $\mathbb{R}^d$, $\Sigma = supp(\mu)$. For a fixed integer $n$, $n \leq d$, define for $x \in \Sigma$, $r > 0$ and $t \in (0,1]$

$$R_t(x,r) = \frac{\mu(B_{tr}(x))}{\mu(B_r(x))} - t^n$$

which encodes the doubling properties of $\mu$. We say $\mu$ is $n$-asymptotically optimally doubling ($n$-AOD) if for each compact set $K \subset \mathbb{R}^d$, $x \in K$ and $t \in [0,1]$, we have

$$\lim_{r \to 0^+} \sup_{x \in K} |R_t(x,r)| = 0$$

**Theorem (Kenig-Toro)**

*Let $\mu$ be a Radon measure in $\mathbb{R}^d$ that is doubling and n-asymptotically optimally doubling. Then all pseudo-tangent measures of $\mu$ are n-uniform.*

## Definition

Let $\mu$ be a Radon doubling measure in $\mathbb{R}^d$, $\Sigma = spt(\mu)$. We say $\mu$ is uniformly asymptotically doubling (UAD) if there exists a continuous function $f_\mu : \Sigma \times \mathbb{R}_+ \to \mathbb{R}_+$, $f_\mu(x, 1) = 1$ for every $x \in \Sigma$ such that, for every $K$ compact with $K \cap \Sigma \neq \emptyset$:

$$\lim_{r \to 0} \sup_{x \in K} \left| \frac{\mu(B_{tr}(x))}{\mu(B_r(x))} - f_\mu(x, t) \right| = 0, \text{ for } x \in K \cap \Sigma, \ t \in (0, 1].$$

We call $f_\mu$ the distribution function associated to $\mu$.

**Theorem (N.,'18)**

*Let $\mu$ be a uniformly asymptotically doubling measure in $\mathbb{R}^d$. Then all pseudo-tangents of $\mu$ are uniform. More precisely, if $\xi \in supp(\mu)$, and $\nu$ is a pseudo-tangent to $\mu$ at $\xi$, then for every $x \in supp(\nu)$, and every $r > 0$ we have :*

$$\nu(B_r(x)) = f_\mu(\xi, r).$$

**Lemma (N.,18, direct consequence of (Preiss))**

*Let $\mu$ be a Uniformly Asymptotically Doubling measure and $f$ be its distribution function. Then for every $x$ there exists $n = n_x$ such that:*

$$\lim_{t\to 0} \frac{f(x,t)}{t^n} = f(x),$$

*where $f(x) \in (0,\infty)$. We say $\mu$ is n-UAD for $n = \max_x n_x$.*

### Theorem (N., 2018)

*Let $\mu$ be a n-UAD measure in $\mathbb{R}^d$, $3 \leq n \leq d$. Then*

$$dim_{\mathcal{H}}(\mathcal{S}_\nu) \leq n - 3.$$

Thank you!