Characterization of Branched Covers with Simplicial Branch Sets

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Branched Covers

Defintion

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- At most points f is a local homeomorphism. The *branch set* of f, denoted B_f, is the set of points where f fails to be a local homeomorphism.
- Branched covers are topological generalization of quasiregular maps.

Branched Covers in Dimension Two

In two dimensions the typical example of a branched cover is a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.

- The branch set is the finite set of critical points of f.
- Near the branch points, f behaves like the map z^d, where d is the degree of the critical point.

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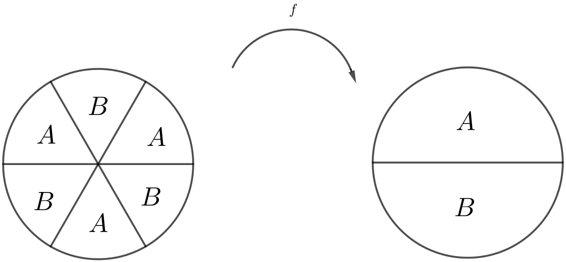
- Near the branch points, f behaves like the map z^d, where d is the degree of the critical point.
- Topologically, this map is equivalent to a winding map: $(r, \theta) \mapsto (r, d\theta)$.



Up to homeomorphism, this characterizes every branched cover.

Theorem (Stoïlow)

Let $f\colon S^2 \to \widehat{\mathbb{C}}$ be a branched cover. Then there exists a homeomorphism $h\colon \widehat{\mathbb{C}} \to S^2$ so that $f \circ h$ is a rational map.

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Corollary

Every branched cover from $S^2 \to S^2$ is equivalent up to a homeomorphism to a piecewise linear (PL) map.

Motivation

Definition

A map $f : \Omega \to \mathbb{R}^n$ is *K-quasiregular* if $f \in W^{1,n}_{\text{loc}}(\Omega)$ and for almost every $x \in \Omega$,

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- By a theorem due to Reshetnyak, quasiregular maps are branched covers.
- The converse is generally false and it is difficult to construct quasiregular maps.
- PL maps are generally quasiregular.

Higher Dimensions

Theorem (Church and Hemmingsen, '60)

Let $f : \Omega \to \mathbb{R}^n$ *be a branched cover, where* Ω *is a domain in* $\mathbb{R}^n$*. If* $f(B_f)$ *can be embedded into a codimension 2 plane, then* f *is topologically equivalent to a winding map.*

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- By a theorem due to Černavskiĭ and Väisälä, B_f and $f(B_f)$ have topological dimension less than $n-2$.
- In dimension 2 this hypothesis is always satisfied, but it is not always satisfied in higher dimensions.

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  is a branched cover with branch set $B \simeq S^1$.
- Note that $\pi_1(S^5 \setminus B)$ has order 120.

Generalizing Church and Hemmingsen

Theorem (Martio and Srebro, '79)

Let $f\colon \Omega \to \mathbb{R}^3$ be a branched cover and $x_0 \in B_f$. If there exists and open neighborhood V of x_0 so that the image of the branch set $f(B_f \cap V)$ can be embedded into a union of finitely many line segments originating from $f(x_0)$, then f is topologically equivalent on V to a cone of a rational map $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.

- A cone of a map g is the map

$$g \times \mathrm{id}\colon\ \mathrm{cone}(\widehat{\mathbb{C}}) \to \mathrm{cone}(\widehat{\mathbb{C}}),$$

$$\mathrm{cone}(\widehat{\mathbb{C}}) = \frac{\widehat{\mathbb{C}} \times [0,1]}{\{(z,0) \sim (w,0)\}}$$

($\widehat{\mathbb{C}} \times [0,1]$ with this identification is homeomorphic to B^3).

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- This implies that f is topologically equivalent to a PL map.

Theorem (Luisto and P., '18)

Let $f : \Omega \to \mathbb{R}^n$ be a branched cover and $x_0 \in B_f$. If there exists an open neighborhood V of x_0 so that the image of the branch set $f(B_f \cap V)$ can be embedded into an $(n-2)$-simplicial complex, then f is topologically equivalent on V to a cone of a PL map $g : S^{n-1} \to S^{n-1}$.

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Main Result

Theorem (Luisto and P., '18)

Let $f\colon \Omega \to \mathbb{R}^n$ be a branched cover and $x_0 \in B_f$. If there exists an open neighborhood V of x_0 so that the image of the branch set $f(B_f \cap V)$ can be embedded into an $(n-2)$-simplicial complex, then f is topologically equivalent on V to a cone of a PL map $g\colon S^{n-1} \to S^{n-1}$.

- This implies that f is topologically equivalent to a PL map.
- This theorem also extends to a global result for $f\colon S^n \to S^n$.

Main Result

We can use this result to construct quasiregular maps.

Corollary

For each $n \in \mathbb{N}$ there exists a quasiregular map $f : \mathbb{R}^{2n} \to \mathbb{CP}^n$.

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For each $n \in \mathbb{N}$ there exists a quasiregular map $f\colon \mathbb{R}^{2n} \to \mathbb{CP}^n$.

- The map

$$([z_1 : w_1], \cdots, [z_n : w_n]) \mapsto$$
$$[z_1 \cdots z_n : \sum_{i=1}^{n} z_1 \cdots \widehat{z_i} \cdots z_n w_i : \cdots : w_1 \cdots w_n]$$

is a branched cover from $\mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1 \to \mathbb{CP}^n$.

Normal Neighborhoods

Theorem (Luisto and P., '18)

Let $f\colon \Omega \to \mathbb{R}^n$ be a branched cover and $x_0 \in B_f$. If there exists an open neighborhood V of x_0 so that the image of the branch set $f(B_f \cap V)$ can be embedded into an $(n-2)$-simplicial complex, then f is topologically equivalent on V to a cone of a PL map $g\colon S^{n-1} \to S^{n-1}$.

Let $f\colon \Omega \to \mathbb{R}^n$ be a branched cover and $x_0 \in \Omega$ be a point. There exists a radius $r_0 > 0$ and a family of neighborhoods, denoted $U(x_0, r)$, such that for $0 < r \leq r_0$

- $x_0 \in U(x_0, r)$
- $f(U(x_0, r)) = B(f(x_0), r)$
- $f(\partial U(x_0, r) = \partial B(f(x_0), r)$
- $f^{-1}\{f(x_0)\} \cap U(x_0, r) = \{x_0\}$

Outline of Proof

- Suppose that near x_0, $\partial U(x_0, r)$ is homeomorphic to S^{n-1}.
- It is a fact that restricted to $\partial U(x_0, r)$, f is still a branched cover. So if we induct on the dimension, $f\colon \partial U(x_0, r) \to S^{n-1}$ is equivalent to a PL map.

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- Then by a path lifting argument we show that f behaves the same way topologically on the boundaries of $U(x_0, r)$ for all sufficiently small r.
- So f is equivalent to a cone of a PL map.
- It is not clear that $\partial U(x_0, r) \simeq S^{n-1}$, in fact it may not even be a manifold.

- In dimension two, f is locally injective on $\partial U(x_0, r)$ and so $\partial U(x_0, r)$ is a manifold and therefore is homeomorphic to S^1.

Back to Dimensions Two and Three

- In dimension two, f is locally injective on $\partial U(x_0, r)$ and so $\partial U(x_0, r)$ is a manifold and therefore is homeomorphic to S^1.
- In dimension three Martio and Srebro show that $\partial U(x_0, r) \simeq S^2$.
- $\partial U(x_0, r)$ is a manifold using a similar argument as in dimension two.
- $U(x_0, r)$ is contractible so $\partial U(x_0, r) \simeq S^2$.

$\partial U(x_0, r)$ is a Manifold

- f restricted to $\partial U(x_0, r)$ is a branched cover and away from the branch set is a covering map.
- So $\partial U(x_0, r) \setminus B_f$ is a manifold.
- If $x \in \partial U(x_0, r) \cap B_f$, then we consider the map f restricted to a normal neighborhood of x in $\partial U(x_0, r)$.
- We continue this way to go down in dimension considering more and more nested normal neighborhoods.

Normal Neighborhoods

There is a partial converse to the Martio-Srebro result.

Theorem (Martio and Srebro, '79)

Let $f : \Omega \to \mathbb{R}^3$ *be a branched cover so that at* $x \in \Omega$ *there exists an* $r_0 > 0$ *with the property that for all* $r \leq r_0$, $\partial U(x_0, r)$ *is a manifold. Then at* x_0, f *is equivalent to a path of rational maps.*

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We show a corresponding result:

Theorem (Luisto and P, '18)

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Thank you!