Excess-height bound, with application to a harmonic measure free boundary problem

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We say a set $E \subset \mathbb{R}^n$ is a **set of locally finite perimeter**, if

$$\sup\left\{\int_E \operatorname{div}\varphi dX : \varphi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n) \text{ and } \|\varphi\|_\infty \leq 1\right\} < +\infty.$$

It then follows that

$$\int_E \operatorname{div}\varphi dX = -\int_{\partial^* E} \varphi \cdot \nu_E d\mathcal{H}^{n-1}, \quad \text{for any } \varphi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n),$$

where $\partial^* E \subset \partial E$ is called the reduced boundary of E, ν_E the unit normal vector to E and $|\nu_E| = 1$ a.e..

We say the vector-valued Radon measure $\mu_E := \nu_E \mathcal{H}^{n-1}|_{\partial^* E}$ is the Gauss-Green measure of E. Note that $|\mu_E| = \mathcal{H}^{n-1}|_{\partial^* E}$ is the perimeter measure of E.

Proposition (Excess-height bound)

Suppose E is a set of locally finite perimeter such that $\partial E = \operatorname{spt}\mu_E$ and $|\mu_E|$ is Ahlfors regular. There exists $\epsilon > 0$ such that if the excess

$$e_n(x_0, 2r) := \frac{1}{r^{n-1}} \int_{C(x_0,2r,e_n)\cap\partial^* E} \frac{|\nu_E - e_n|^2}{2} d\mathcal{H}^{n-1} < \epsilon$$

for $x_0 \in \partial E$ and $r > 0$, then

$$\frac{1}{r} \sup\left\{|q(x) - q(x_0)| : x \in C(x_0, r, e_n) \cap \partial E\right\} \leq Ce_n(x_0, 2r)^{\frac{1}{2(n-1)}}.$$

Moreover, there exists a Lipschitz function $u : \mathbb{R}^{n-1} \to \mathbb{R}$ with $\operatorname{Lip} u \leq 1$ such that

$$\frac{\mathcal{H}^{n-1}\left(M \Delta \operatorname{Gr}(u)\right)}{r^{n-1}} \leq Ce_n(x_0, r), \quad \text{where } M := C(x_0, r, e_n) \cap \partial E.$$

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Remark

This type of excess-height bound has been known for a long time if the set E minimizes the perimeter. This bound was the first step towards proving the regularity of perimeter minimizers. (c.f. De Giorgi)

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Remark

Roughly speaking, it says if the unit normal of a domain Ω has small oscillation (bounded by ϵ), its boundary is $\epsilon^{\frac{1}{n-1}}$ Reifenberg-flat; moreover Ω and Ω^c is *well-separated* by the strip.

WLOG we may assume $x_0 = 0$ and $r = 1$.

Lemma (Separation lemma)

Given $t_0 \in (0,1)$, *there exists* $\delta = \delta(t_0)$ *such that if* $e_n(0,2) < \delta$, *then*

$$|q(x)| < t_0, \quad \text{for any } x \in M,$$

and

$$|\{x \in C(0,1,e_n) \cap E : q(x) < -t_0\}| = 0,$$
$$|\{x \in C(0,1,e_n) \cap E^c : q(x) > t_0\}| = 0,$$

Proposition (Compactness for sets of finite perimeter)

Suppose $\{E_k\}$ is a sequence of sets of locally finite perimeter whose boundaries are Ahlfors regular with uniform constants and $\partial E_k = \operatorname{spt} \mu_{E_k}$. Assume $0 \in \partial E_k$, then passing to a subsequence there are a set of locally finite perimeter E and a Radon measure μ such that

$$E_k \to E \text{ in } L^1, \quad \mu_{E_k} \rightharpoonup \mu_E, \quad |\mu_{E_k}| \rightharpoonup \mu.$$

Moreover, μ is Ahlfors regular, $|\mu_E| \le \mu$, and

1. *If $x \in \partial E$, then there exist $x_k \in \partial E_k$ such that $x_k \to x$;*
2. *If $x_k \in \partial E_k$ and $x_k \to x$, then $x \in \operatorname{spt} \mu$.*

Proof of the proposition

Proof by pictures.

Consider the function $f : (-1, 1) \to [0, \mathcal{H}^{n-1}(M)]$ defined by

$$f(t) = \mathcal{H}^{n-1}(M \cap \{q(x) > t\}).$$

Recall that the excess bounds the difference between this measure and the measure of its projection onto $\mathbb{R}^{n-1}$.

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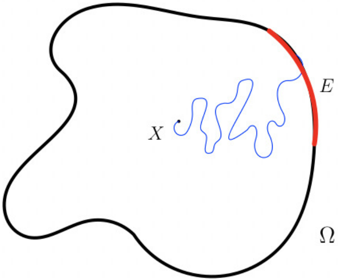
We denote the slicings $E_t = \{z \in \mathbb{R}^{n-1} : (z, t) \in E\}$ and $(\partial^* E)_t = \{z \in \mathbb{R}^{n-1} : (z, t) \in \partial^* E\}$.
By the co-area formula

$$\int_{\mathbb{R}} \int_{(\partial^* E)_t} g \, d\mathcal{H}^{n-2} dt = \int_{\partial^* E} g \sqrt{1 - (\nu_E \cdot e_n)^2} \, d\mathcal{H}^{n-1}$$

Application to harmonic measure



For any $E \subset \partial\Omega$, its **harmonic measure** is

$$\omega(E) := \mathbb{P}\left(\text{Brownian motion } B_t^X \text{ exits the domain } \Omega \text{ from } E\right).$$

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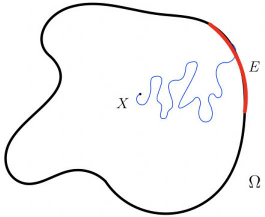

Consider the Dirichlet boundary value problem for the Laplacian

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega \\ u = f, & \text{on } \partial\Omega. \end{cases}$$

The **harmonic measure** is the unique measure such that

$$u(X) = \int_{\partial\Omega} f \; d\omega^X.$$

One expects that the harmonic measures for *nice domains* have good behavior, e.g. $\omega \ll \sigma := \mathcal{H}^{n-1}|_{\partial\Omega}$.
See the work of Dahlberg, David-Jerrison, Semmes, Hofmann-Martell, etc.

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The type of problem we consider is the converse:

Harmonic measure free boundary problem

Suppose the harmonic measure has nice behavior (for example, suppose $\log \frac{d\omega}{d\sigma} \in \mathrm{VMO}$), what can we deduce about the domain?

For previous results in this direction, see Jerison, Kenig-Toro, Azzam-Tolsa-Mourgoglou etc. and Azzam-Hofmann-Martell-Mayboroda-Mourgoglou-Tolsa-Volberg.

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Our approach is:

$$\log \frac{d\omega}{d\sigma} \in \mathrm{VMO} \xRightarrow{\textit{singular integral}} \nu \in \mathrm{VMO} \xRightarrow{\textit{excess}-\textit{height bound}} \Omega \text{ is nice.}$$

$$\uparrow$$

$$\lim_{\substack{Z \to x \\ Z \in \Gamma(x)}} \nabla \mathcal{S} f(Z) = \frac{1}{2}\nu(x) f(x) + \mathcal{T} f(x)$$

Thank you!