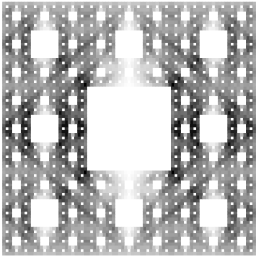
Conformal Dimension via p-resistance: Sierpiński Carpet

Topics at the interface of analysis and geometry

AMS Meeting, Hawaii

(22 March 2019)



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Q: Why Montana? A: 20 min by car + lift/hike time (from desk):

Conformal Dimension

[All spaces Ahlfors regular.]

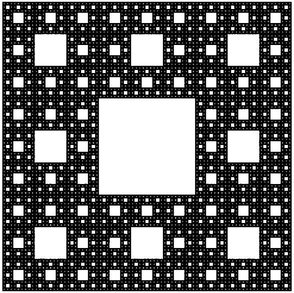
Definition (Bourdon, Pajot (2003))

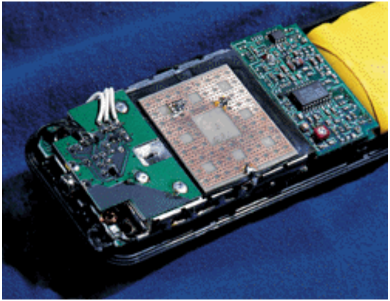
(Ahlfors-regular) **conformal dimension** *of* X *is*

$$\dim_{conf}(X) := \inf \{\dim_{HD}(Y) : \ \exists f : X \to Y \ q.s. \ map\}$$

If **inf** *is attained by* f_{opt}, *call* $f_{opt} : X \to Y_{opt}$ *a* **quasi-symmetric uniformization** *of* X.

Today: X = Sierpinski Carpet (but it all generalizes!)





$$\dim_{HD}(X) = \ln 8 / \ln 3 \approx 1.893$$

Digression: Round Carpets of Geometric Origin

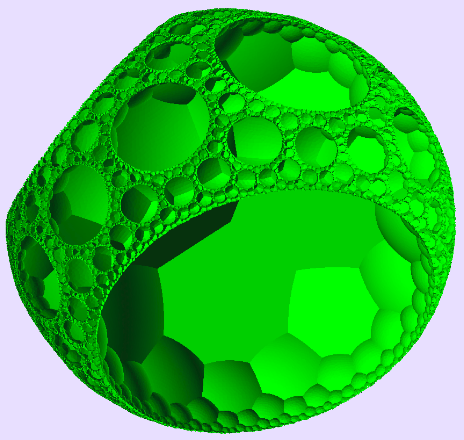


Figure: Sierpinski Schläfi-$(7, 3, 3)$ honeycomb (by Danny Calegari); 7-gon faces; 3 meeting a vertex; 3 polyhedra along an edge.

SOURCE: Danny Calegari's page at https://math.uchicago.edu/~dannyc/gallery.html

The Problem

$$\dim_{\mathrm{conf}}(X) = ?$$

$$f_{\mathrm{opt}} : X \to Y = ???$$

New (2016) Results

- "Convincing" numerical prediction

$$\dim_{\text{conf}}(X) \approx 1.7967$$

  c.f. R. Malo (2015 PhD under L. Geyer): $1.7 \leq \dim_{\text{conf}}(X) \leq 1.8$

- upper/lower bounds by hand:

$$1.704 < \dim_{\text{conf}}(X) < 1.848$$

  c.f. Keith and Laakso (2004) and Kigami (2010)

$$1.6309 \approx 1 + \frac{\ln 2}{\ln 3} \leq \dim_{\text{conf}}(X) \leq \frac{\ln\left((9+\sqrt{41})/2\right)}{\ln 3} \approx 1.858$$

- method for computer assisted bounds of arbitrary precision
- conjectural construction of uniformization by a slit carpet

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- method for computer assisted bounds of arbitrary precision
- conjectural construction of uniformization by a slit carpet
- sup- and sub- multiplicativity of modulus/resistance
- two sided modulus/resistance bounds at criticality

Conformal Dimension $\equiv$ Resistance Dimension

NEW/OLD IDEA: Use electrical p-currents to probe X

Theorem

For the carpet (and, likely, any half-decent X): *If $\mathcal{G}_n$ is a* **resistor network** *approximating X and $R_n(\mathrm{p})$ is its* p-**resistance**, *then* $\dim_{conf}(X)$ *is the* **critical exponent**

$$\mathrm{p}_{res} := \inf\{\mathrm{p} > 1 : \limsup_{n\to\infty} R_n(\mathrm{p}) = \infty\}$$

`I will explain the terms soon but:`

- Proof of Th: p-**resistance** $\sim$ p-**extremal length** ...
- easy via scheme of Carrasco Piaggio (after Pansu, Bourdon-Kleiner,...) instantiated by Geyer-Malo for X

But the **devil** is elsewhere: **engineering** networks with good estimates.

Key Numerical Graph: p-resistance vs p

$$\dim_{\mathrm{conf}}(X) = \mathrm{p}_{res} = \text{ the } \textbf{pivot point:}$$

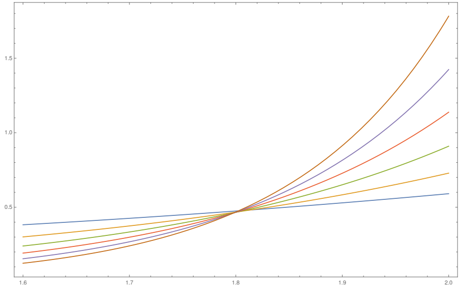


Figure: Resistance $R_n(\mathrm{p})$, over $1.6 \leq \mathrm{p} \leq 2$, for approximation resolution 3^{-n} with $n = 1, ..., 6$

DC circuit $\equiv$ resistor network

- **network** $\equiv$ di-graph $(\mathcal{V}, \mathcal{E})$ with weights $r : \mathcal{E} \to [0, \infty]$
- **marked network** has selected $\mathcal{A}, \mathcal{B} \subset \mathcal{V}$ **input** and **output** vertices ($\mathcal{A} \cap \mathcal{B} = \emptyset$)

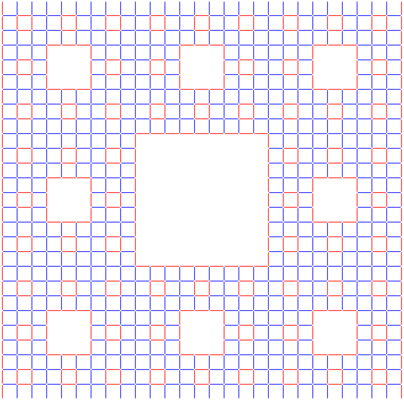


Figure: $\mathcal{G}_3$; $r = 0.5$ blue, $r = 1$ red edges; $\mathcal{A} =$ top; $\mathcal{B} =$ bottom.

By The Way:

[optional]

Q: Why uneven (1 or 0.5) resistance?

A: Of course, Growing Networks by Substitution:

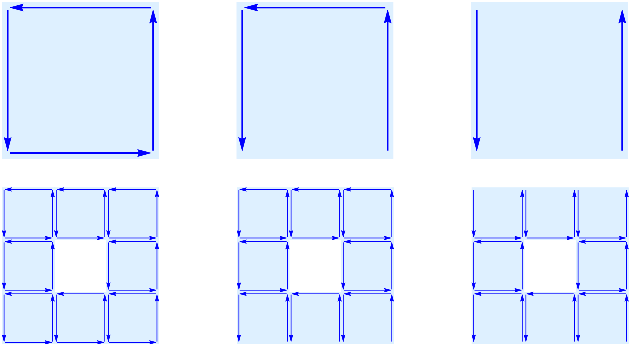


Figure: All unit resistors. (Vertices suppressed.)

p-currents and p-resistance (of a marked network)

$1/\mathrm{p} + 1/\mathrm{q} = 1, \ (\mathrm{p}, \mathrm{q} > 1)$ [**Note**: $\mathrm{p}/\mathrm{q} = \mathrm{p} - 1, \ \mathrm{q}/\mathrm{p} = \mathrm{q} - 1$]

- p-**power** of a **flow** $\mathcal{J}$ is the weighted q-norm:

$$P(\mathcal{J}) := \sum_{e \in \mathcal{E}} (r(e)^{1/\mathrm{p}} |\mathcal{J}(e)|)^{\mathrm{q}}$$

[DEF: $\mathcal{J}$ is a **flow** $\equiv$ $\mathcal{J}$ satisfies **1st Kirchhoff Law**]

- p-**resistance** (from $\mathcal{A}$ to $\mathcal{B}$) is R given by

$$R^{1/\mathrm{p}} := \min_{\mathcal{J}} \frac{P(\mathcal{J})^{1/\mathrm{q}}}{I(\mathcal{J})}$$

over **flows** $\mathcal{J}$ (from $\mathcal{A}$ to $\mathcal{B}$) with **flux** $I(\mathcal{J}) = I \neq 0$.

Potential ($\equiv$ Lagrange Multipliers)

Fix flux $I \neq 0$ and introduce **Lagrange multipliers**:

- $\mathcal{U}(v) \in \mathbb{R}$ enforcing $\operatorname{div}\mathcal{J}(v) = 0$ (at each $v \notin \mathcal{A} \cup \mathcal{B}$)
- $U \in \mathbb{R}$ enforcing **flux** $\sum_{v \in \mathcal{B}} \operatorname{div}\mathcal{J}(v) = I$

Recast multipliers as **potential**

$$\mathcal{U} : \mathcal{V} \to \mathbb{R}, \quad \mathcal{U}|_{\mathcal{A}} = 0,\ \mathcal{U}|_{\mathcal{B}} = U$$

The **Lagrangian** (after summation by parts):

$$\mathcal{L} \;=\; \sum_{e \in \mathcal{E}} \frac{r(e)^{\mathtt{q}/\mathtt{p}}}{\mathtt{q}} |\mathcal{J}(e)|^{\mathtt{q}} - \nabla\mathcal{U}(e)\mathcal{J}(e) + UI$$

p-Ohm Law ($\equiv$ Euler-Lagrange Equations)

From **Lagrangian** $\mathcal{L}(\mathcal{J},\mathcal{U}) = \sum_{e\in\mathcal{E}} \frac{r(e)^{\mathrm{q}/\mathrm{p}}}{\mathrm{q}} |\mathcal{J}(e)|^{\mathrm{q}} - \nabla\mathcal{U}(e)\mathcal{J}(e)$

Euler-Lagrange eqn (apart from **constraints**) give

- p-**Ohm Law** (for each edge) Use $\mathrm{q}-1=\mathrm{q}/\mathrm{p}$

$$(\nabla\mathcal{U}(e))^{\mathrm{p}} = r(e)^{\mathrm{q}}(\mathcal{J}(e))^{\mathrm{q}} \qquad \left(\equiv \frac{\partial\mathcal{L}}{\partial\mathcal{J}(\cdot)} = 0\right)$$

`Notation:` $(s)^{\mathrm{p}} := \mathrm{sgn}(s)|s|^{\mathrm{p}}$

- p-**current** is a flow $\mathcal{J}$ satisfying p-Ohm for some $\mathcal{U}:\mathcal{V}\to\mathbb{R}$
- p-**potential** is $\mathcal{U}:\mathcal{V}\to\mathbb{R}$ whose $\mathcal{J}$ satisfies 1st Kirchhoff Law

p-**Laplace Eq:** $$\mathrm{div}\left(\frac{\nabla\mathcal{U}^{\mathrm{p}/\mathrm{q}}}{r}\right) = 0$$

Lagrange-Hölder Duality

p-**power of potential** $\mathcal{U} : \mathcal{V} \to \mathbb{R}$ is a weighted p-norm:

$$P(\mathcal{U}) := \sum_e \left(\frac{|\nabla \mathcal{U}(e)|}{r(e)^{1/\mathrm{p}}} \right)^{\mathrm{p}}$$

Hölder inequality (for a flow $\mathcal{J}$ with **flux** I and $\mathcal{U}$ with **drop** U):

$$UI = \sum_e \nabla \mathcal{U}(e) \mathcal{J}(e) \leq \quad P(\mathcal{J})^{1/\mathrm{q}} P(\mathcal{U})^{1/\mathrm{p}}$$

rewrites as (error bounding!) **duality gap**:

$$\frac{U}{P(\mathcal{U})^{1/\mathrm{p}}} \leq R^{1/\mathrm{p}} \leq \frac{P(\mathcal{J})^{1/\mathrm{q}}}{I}$$

At the optimum **the gap closes**:

$$\max_{\mathcal{U}} \frac{U}{P(\mathcal{U})^{1/\mathrm{p}}} = R^{1/\mathrm{p}} = \min_{\mathcal{J}} \frac{P(\mathcal{J})^{1/\mathrm{q}}}{I}$$

Summary of p-circuit Theory

dual and **primal** optimization problems:

$$\max_{\mathcal{U}} \frac{U}{P(\mathcal{U})^{1/\mathrm{p}}} = R^{1/\mathrm{p}} = \min_{\mathcal{J}} \frac{P(\mathcal{J})^{1/\mathrm{q}}}{I},$$

At optimum:

- p-**Ohm Law** links current and potential:

$$(\nabla \mathcal{U})^{\mathrm{p}} = (r\mathcal{J})^{\mathrm{q}} \qquad \text{and} \qquad U^{\mathrm{p}} = (RI)^{\mathrm{q}}$$

- **Joule's Law** gives the dissipated power:

$$P = UI = \frac{U^{\mathrm{p}}}{R} = R^{\mathrm{q}/\mathrm{p}} I^{\mathrm{q}}$$

Off optimum:

- **Error bounds** from **duality gap**:

$$\frac{U}{P(\mathcal{U})^{1/\mathrm{p}}} \le R^{1/\mathrm{p}} \le \frac{P(\mathcal{J})^{1/\mathrm{q}}}{I}$$

Poincare Duality

$\mathcal{G}^*$:= dual of $\mathcal{G}$ with resistance $r^*(e^*)^{1/\mathrm{q}} := r(e)^{-1/\mathrm{p}}$.

Let

$$\mathcal{J}^*(e^*) = \nabla\mathcal{U}(e) \quad \text{and} \quad \nabla\mathcal{U}^*(e^*) = \mathcal{J}(e).$$

Then

$$P^*(\mathcal{J}^*) = P(\mathcal{U}) \quad \text{and} \quad P^*(\mathcal{U}^*) = P(\mathcal{J}).$$

1st Kirchhoff for $\mathcal{J}$ $\leftrightarrow$ **2nd Kirchhoff** for $\nabla\mathcal{U}^*$

2nd Kirchhoff for $\mathcal{U}$ $\leftrightarrow$ **1st Kirchhoff** for $\mathcal{J}^*$

p-**Ohm** for $\mathcal{U}$, $\mathcal{J}$ $\leftrightarrow$ q-**Ohm** for $\mathcal{U}^*$, $\mathcal{J}^*$

At optimum:

$$P^* = P \quad \text{and} \quad R^{1/\mathrm{p}} R^{*1/\mathrm{q}} = 1$$

Poincaré Duality: Simple Example

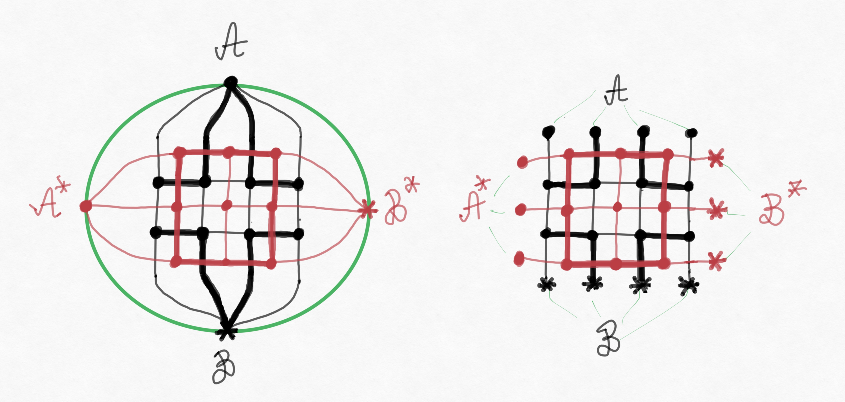


Figure: Two (equivalent) pairs, $(\mathcal{G}, \mathcal{A}, \mathcal{B})$ and $(\mathcal{G}^*, \mathcal{A}^*, \mathcal{B}^*)$, of topologically dual marked networks. (The one on the left is embedded in a closed disk bounded by the green circle.) Thick black edges have resistance 2^{-1}; thick red $2^{\mathrm{q}/\mathrm{p}}$; all other resistances are unit.

Poincaré Dual Network with Superconducting Islands

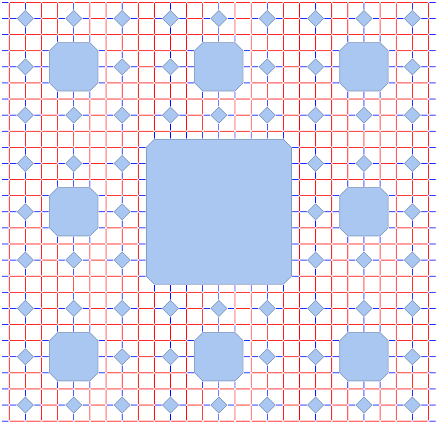


Figure: Poincaré Dual Network $\mathcal{G}_3^*$: edge resistance is $r(e) = 2^{p-1}$ for blue and $r(e) = 1$ for red edges; the blue islands are superconducting.

Two dualities = four optimizations $\Longrightarrow$ happy computing!

@ $\mathtt{p} \approx 1.8$ and $\mathtt{q} \approx 2.4$

- $\min_{\mathcal{U}} \left(P(\mathcal{U}) = \sum |\nabla \mathcal{U}|^{\mathtt{p}} \right)$: non-C^2, no vertex constraints
- $\min_{\mathcal{J}} \left(P(\mathcal{J}) = \sum |\mathcal{J}|^{\mathtt{q}} \right)$: C^2, vertex constraints
- $\min_{\mathcal{J}^*} \left(P^*(\mathcal{J}^*) = \sum |\mathcal{J}^*|^{\mathtt{p}} \right)$: non-C^2, vertex constraints
- $\min_{\mathcal{U}^*} \left(P^*(\mathcal{U}^*) = \sum |\nabla \mathcal{U}^*|^{\mathtt{q}} \right)$: C^2, no vertex constraints

The last approach: least RAM and smooth enough for L-BFGS quasi-Newton to converge in circa 40 h in sub 11 steps for $n = 8$ with 10^{-8} precision in R_n and graph size of order 9^8.
(L-BFGS = limited memory Broyden-Fletcher-Goldfarb-Shanno.)

Poincaré duality and Uniformization

Conjecture: $f_n := (\mathcal{U}_n, \mathcal{U}_n^*)$ converge to q.s. f_{opt} onto **slit carpet**:

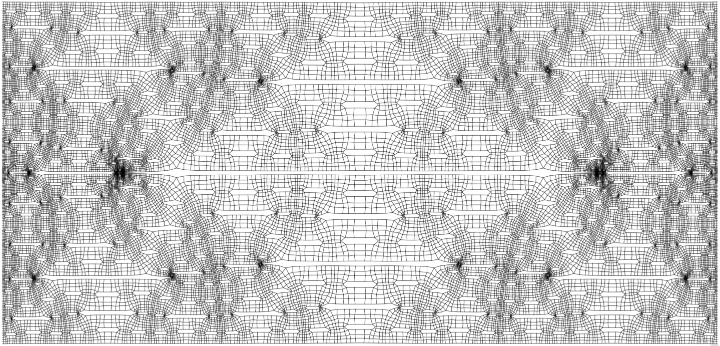


Figure: Approximate Uniformized Carpet ($n = 5$)

NOTE: Less visually appealing but more effective: define d_{opt} by using the dissipated heat as weight.

Theoretical benefits of Poincaré duality

- $R^{1/\mathrm{p}} \leq \frac{P(\mathcal{J})^{1/\mathrm{q}}}{I}$ given **any** flow $\mathcal{J}$
- $R^{1/\mathrm{p}} = R^{*\,-1/\mathrm{q}} \geq \left(\frac{P^*(\mathcal{J}^*)^{1/\mathrm{q}}}{I}\right)^{-1}$ given **any** flow $\mathcal{J}^*$

Theorem

There are $\alpha, \beta > 0$ such that **sup-/sub-mult** *holds:*

$$\alpha^{\mathrm{p}} R_n(\mathrm{p}) R_m(\mathrm{p}) \leq R_{n+m}(\mathrm{p}) \leq \beta^{\mathrm{p}} R_n(\mathrm{p}) R_m(\mathrm{p}).$$

Corollary

Upper/Lower Bounds on p_{res} (e.g. via exact knowledge of α, β).

Proofs: Construct good flows via **self-similarity** for all $n \in \mathbb{N}$

Proof of sup-mult/upper bound (sub-mult/lbd similar)

MAIN TASK:

Construct a flow $\mathcal{J}^*_{n+m}$ (on $\mathcal{G}^*_{n+m}$) from currents $\mathcal{J}^*_n$ and $\mathcal{J}^*_m$ by gluing **projective flo-tiles**. [Ditto for $\mathcal{J}_{n+m}$ but a bit harder...]

KEY ISSUES: Matching the trans boundary flows

... and power control

Sup-multiplicativity/upper bound Substitution

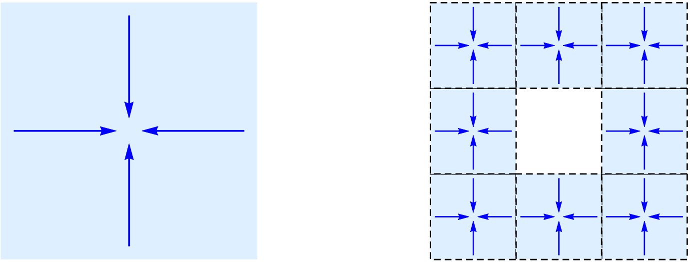
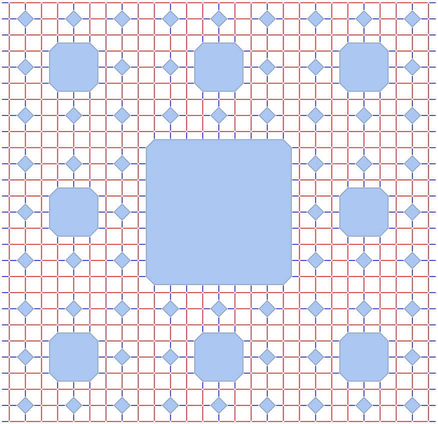


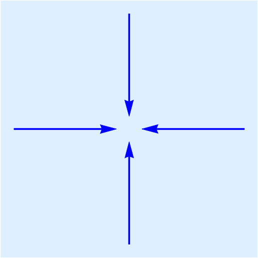
Figure: Tile substitution rule $\Phi : T_{\text{ini}}$ (left) $\mapsto \Phi(T_{\text{ini}})$ (right).

... grows the Network with Superconducting Islands



but still have to design flows ...

Mixed Replacement Flows (for sup-mult/upper Bound)



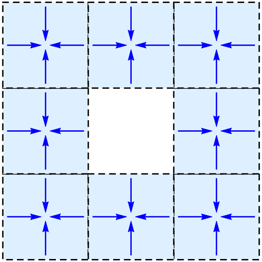


Figure: Flow replacement with prescribed fluxes (x_1, x_2, x_3, x_4)

TO DO: replace **currents** (in $\mathcal{G}_0^*$) by **flows** (in $\mathcal{G}_m^*$): (here $m = 1$)

$$\mathcal{J}^*_{x_1x_2x_3x_4} = t_1\mathcal{J}^*_{1010} + t_2\mathcal{J}^*_{1100} + t_3\mathcal{J}^*_{0101} + t_4\mathcal{J}^*_{0110} + t_5\mathcal{J}^*_{0011} + t_6\mathcal{J}^*_{1001}$$

Optimize **mixture parameters** t_j for each **flux data** (x_1, x_2, x_3, x_4).

Pure Replacement Flows (for sup-mult/upper bound)

... where current J^*_{1010} comes from **optimization** for R^*_1; and $\mathcal{J}^*_{1100}, \mathcal{J}^*_{0101} \ldots$ *"by symmetry"*:

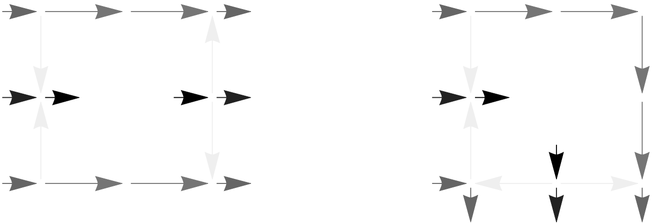


Figure: (Pure) replacement flows $\mathcal{J}^*_{1010}$ and $\mathcal{J}^*_{1100}$.

UPSHOT: Power ratio $\implies$ upper bound (and sup-mult)

By construction: [Here m=1]

$$\frac{{R^*_{n+1}}^{1/\mathrm{p}^*}}{{R^*_n}^{1/\mathrm{p}^*}} \le \rho^* := \max_{(x_1,x_2,x_3,x_4)} \frac{P^*\left(\mathcal{J}^*_{x_1x_2x_3x_4}\right)^{1/\mathrm{p}}}{\left(\sum_{i=1}^4 |x_i|^{\mathrm{p}}\right)^{1/\mathrm{p}}}$$

so PUNCHLINE:

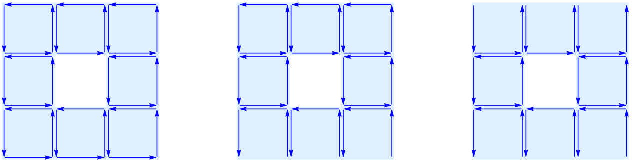
$$\rho^* < 1 \implies \lim_{n\to\infty} R^*_n = 0 \implies \lim_{n\to\infty} R_n = \infty \implies \mathrm{p}_{res} < \mathrm{p}$$

"Q.E.D."

Proof of sub-mult/lower bound

Similar but **harder** games with the standard network ...

... work around ISSUE := resistors on edges of **naïve flo-tiles**



hinders easy flow matching/gluing ...

Substitution Flo-tilings (for sub-mult/lbd) [optional]

The **substitution rule**:

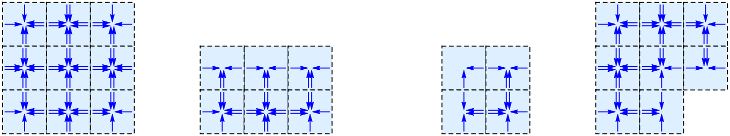


Figure: Sub rule A, B, C, D (above) $\mapsto \Phi(A), \Phi(B), \Phi(C), \Phi(D)$ (below)

Substitution Flo-tiling: super-flo-tiles

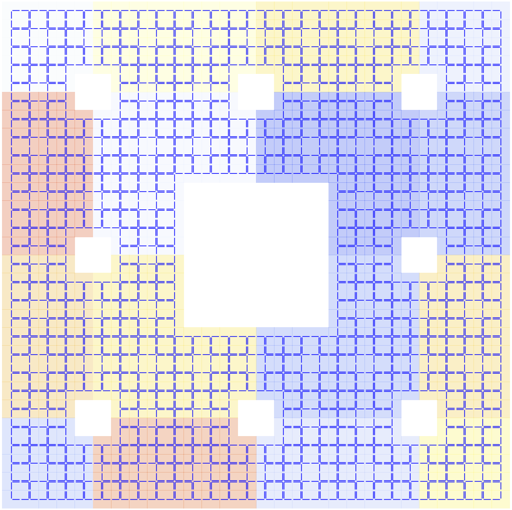


Figure: The network $\mathcal{G}_3$ with the 2-supertiles shaded.

Substitution Flo-tiling: super-flo-tile detail

[optional]

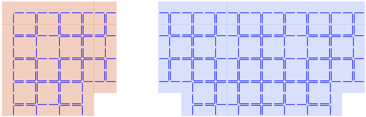


Figure: Any 2-supertile is stitched from several copies of $\mathcal{G}_1^{big}$ and half-$\mathcal{G}_1^{big}$: C 2-supertile (left) uses $1 \times \mathcal{G}_1^{big}$ and $2 \times$ half-$\mathcal{G}_1^{big}$ B 2-supertile (right) uses $2 \times \mathcal{G}_1^{big}$ and $4 \times$ half-$\mathcal{G}_1^{big}$

2nd Substitution for sub-mult/lbd

[optional[2]]

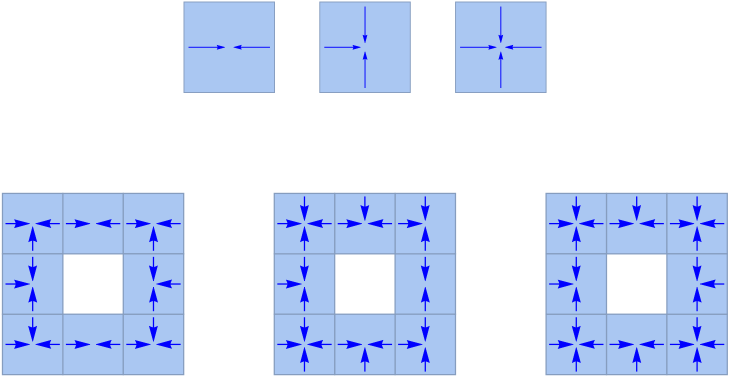
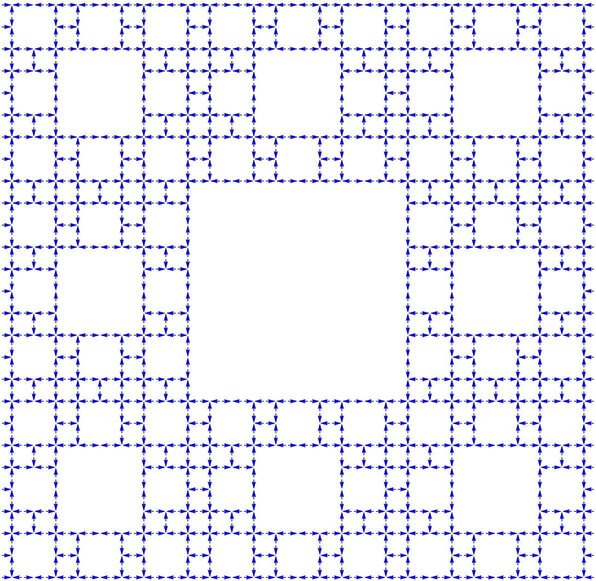


Figure: Sub rule A, B, C, D (above) $\mapsto \Phi(A), \Phi(B), \Phi(C), \Phi(D)$ (below).

2nd Network for sub-mult/lbd

[optional[2]]



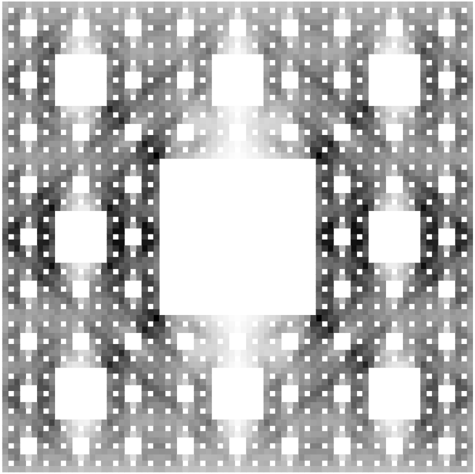
PLAY, CHILD PLAY ...

BUT ... THE REAL FUTURE IS

RENORMALIZATION OPERATOR!

DEF: some other time

Come work with me!



Thank you!