Quantitative Embeddability and Connectivity in Metric Spaces

by

Sylvester David Eriksson-Bique

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Dedication

To my parents, Sirkka-Liisa Eriksson and Stephen Bique.

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Over the years I have accumulated an enormous debt of gratitude to numerous people. This thesis was a result of many unexpected turns, many of which were made possible by the selfless and kind acts of these people. I was fortunate to meet them and I would like to mention here some of them and the ways in which their contributions played a crucial role in this thesis.

The greatest thanks goes to my family. I dedicate this thesis to my parents, Stephen Bique and Sirkka-Liisa Eriksson, who inspired my academic work and shaped the person I am today. They have always stood by me, regardless of my choices. I also thank my sisters Anna-Maria Bique and Linda Eriksson-Bique who I dearly love and admire. I grew up with them and have spent many holidays with them relaxing from work.

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Abstract

This thesis studies three analytic and quantitative questions on doubling metric (measure) spaces. These results are largely independent and will be presented in separate chapters.

The first question concerns representing metric spaces arising from complete Riemannian manifolds in Euclidean space. More precisely, we find bi-Lipschitz embeddings $f \colon A \to \mathbb{R}^N$ for subsets A of complete Riemannian manifolds M^n of dimension n, where N could depend on a bound on the curvature and diameter of A. The main difficulty here is to control the distortion of such embeddings in terms of the curvature of the manifold. In constructing the embeddings, we will study the collapsing theory of manifolds in detail and at multiple scales. Similar techniques give embeddings for subsets of complete Riemannian orbifolds and quotient metric spaces of the form $\mathbb{R}^n/\Gamma$.

The second part of the thesis answers a question about finding quantitative and weak conditions that ensure large families of rectifiable curves connecting pairs of points. These families of rectifiable curves are quantified in terms of Poincaré inequalities. We identify a new quantitative connectivity condition in terms of curve fragments, which is equivalent to possessing a Poincaré inequality with some exponent. The connectivity condition arises naturally in three different contexts, and we present methods to find Poincaré inequalities for the spaces involved. In particular, we prove such inequalities for spaces with weak curvature bounds and thus resolve a question of Tapio Rajala.

In the final part of the thesis we study the local geometry of spaces admitting differentiation of Lipschitz functions with certain Banach space targets. The main result shows that such spaces can be characterized in terms of Poincaré inequal-

ities and doubling conditions. In fact, such spaces can be covered by countably many pieces, each of which is an isometric subset of a doubling metric measure space admitting a Poincaré inequality. In proving this, we will find a new way to use hyperbolic fillings to enlarge certain sub-sets into spaces admitting Poincaré inequalities.

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Chapter 1

Introduction

1.1 General questions

Classical analysis has developed many powerful tools and theorems related to the geometry and analysis on Euclidean spaces. We study generalizations of such tools for metric spaces (X, d) and metric measure space (X, d, μ). Our understanding of "geometry" will include some properties that depend on the choice of a measure μ. The goal of such work is to understand how different analytic results depend on the geometry of Euclidean space, and further to find hidden structure in new contexts. Such research has been called "Analysis on Metric spaces" and has seen much activity over the past few decades. Cf. the recent surveys $[16, 72, 88, 132]$.

This thesis consists of sections addressing three particular questions. The common theme of these questions is to find quantitative conditions on spaces that ensure some similarity between the space and classical Euclidean space $\mathbb{R}^n$. We present first these questions in an intuitive and somewhat overly general form. Following this, we introduce the necessary terminology and state the specific results

we obtain.

The first question asks, when a class of spaces can be expressed in terms of Euclidean space. The particular class we study is that of Riemannian manifolds. In a way, our statement can be thought of as a quantitative Lipschitz category version of Nash embedding theorem.

Question 1 When can we represent a Riemannian manifold M as a subset of $\mathbb{R}^n$ by distorting the distance function by at most a constant factor?

It turns out that many examples of interesting spaces can not be embedded into Euclidean space. We give two examples.

Example 1 Consider the Heisenberg group, which can be represented as the space of upper triangular matrices

$$\mathbb{H} = \left\{ \left. \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \right| a, b, c \in \mathbb{R} \right\}. \tag{1.1.1}$$

The Lie algebra of such a space can be identified by a subspace of the Lie algebra of invertible matrices $\mathfrak{gl} = M_{3\times 3}$. This subspace is the linear span of the elements:

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.1.2}$$

Note that $[X, Y] = Z$. Define μ as the 3-dimensional Hausdorff-measure on $\mathbb{H}$.

Finally, let H be the sub-bundle of the tangent bundle induced by span$\{X, Y\}$. Pick a positive definite metric on H, and denote the induced norm by $|\cdot|$. Define

$$d(x, y) = \inf \left\{ \int_0^1 |\gamma'(t)|\, dt \,\middle|\, \gamma\colon [0,1] \to \mathbb{H}, \gamma'(t) \in H \text{ for all } t \in [0,1] \right\}. \quad (1.1.3)$$

The space $(\mathbb{H}, d, \mu)$ does not possess a bi-Lipschitz embedding into any Euclidean space. This follows from a theorem of Pansu [115], and an observation of Semmes [131]. See also subsection 2.2.3.

Example 2 Consider the so called Laakso space [92]. Let X arise as a measured Gromov-Hausdorff-limit of the following construction. Let $X_0 = [0, 1]$. Define X_1 by taking two copies of X_0 and gluing them along the point 2^{-1}. Define $\pi_1\colon X_1 \to X_0$ to be a projection given by identifying corresponding points on each half. Define X_i and maps $\pi_i\colon X_i \to X_0$ inductively by gluing two copies of X_{i-1} along the corresponding points $\pi_{i-1}^{-1}(k2^{-i})$, where k is an odd number. The space X_i will contain 2^i copies of X_0 glued at a finite set of points. The map π_i maps each copy to X_0. Define μ to be the sum of the Lebesgue measures on each of the copies of X_0 divided by 2^i. This space turns out to accept a Poincaré-inequality and a Rademachers theorem in much the same sense as classical Euclidean space.

While these spaces can not be well-represented as subsets of Euclidean space, they do satisfy two important analytic and measure theoretic inequalities: doubling and a Poincaré-inequality. Metric measure spaces with both of these properties are also called *PI-spaces* or *Poincaré-spaces*.

Doubling is a condition describing finite dimensionality of general metric spaces

and metric measure spaces. See Definitions 1.2.13 and 1.2.12 for the precise formulations. The notion of doubling arises from classical work on spaces of homogeneous type and the work on generalizing harmonic analysis from Euclidean spaces [40, 41, 135]. The class of doubling metric measure spaces is quite large. For example, one can take any singular doubling measure μ on $\mathbb{R}^n$ and the space $(\mathbb{R}^n, |\cdot|, \mu)$ will provide an example. Such measures have been constructed in great abundance [114]. Also any closed subset $E \subset \mathbb{R}^n$ with its restricted distance possesses a doubling measure supported on it (see [100, 145].

The difference between the metric notion of doubling and the measure notion of doubling is subtle. Note, any doubling metric measure space is also a doubling metric space [40, Chapter III]. Also, by a theorem of Vol'berg and Konyagin [145] (see also [80], and certain details from [100]) any complete doubling metric space has a doubling measure. In fact, by the arguments in [80] the family of such measures can be quite large (cf. [114]). Thus, the doubling property itself does not greatly restrict the geometry of a doubling measure on the space. We remark, that for non-complete spaces a measure can be constructed on the completion, since metric doubling is preserved by taking completions.

The spaces we are interested in also satisfy a Poincaré-inequality. A Poincaré-inequality is an inequality for functions relating macroscopic oscillations of functions to the size of infinitesimal oscillations. Such inequalities are restrictive enough to rule out many uninteresting examples, but also general enough to arise in various contexts. As an example, we remark that any space with a Poincaré-inequality is immediately quasiconvex and uniformly perfect. This precludes many subsets of $\mathbb{R}^n$. Further, some less trivial geometries are also precluded, such as the standard Sierpinski carpet [101].

To give an indication of the central role played by these inequalities we mention a few important publications: the seminal paper of Cheeger on a differentiable structure for general metric spaces [28], the paper of Semmes [130] on relating Poincaré-inequalities to large families of curves and quantitative topology, the result of Heinonen and Koskela on the structure of quasiconformal maps on metric spaces [74], and the result of Kleiner and Bonk on quasisymmetric rigidity of metric two-spheres [17]. Classically, Poincaré-inequalities play a crucial role in areas such as partial differential equations (see e.g. [125]), sobolev spaces [70, 75] and the structure of manifolds with lower Ricci-bounds [29, 30, 42]. In light of these results, Heinonen among many others have asked the following general question [72], which is the central topic of chapter 3.

Question 2 When does a metric measure space (X, d, μ) possess a Poincaré-inequality? Find flexible ways of establishing such inequalities and relate it to the geometry of the underlying space.

This question is further motivated by certain conditions closely related to Poincaré-inequalities. For example, Bate and Li introduce in [12] a class of spaces satisfying "non-homogeneous Poincaré-inequalities". They asked, if indeed such spaces would already be PI-spaces. We have been informed by Jana Björn that similar problems arise when assuming only an Orlicz-Poincaré-inequality for a metric measure space. It is unclear in these contexts if the weaker assumption gives a more general class of metric measure spaces. Even Cheeger in [28, Chapter 15] asks if a notion of (ϵ, δ)-thickly connectedness is equivalent to a Poincaré-inequality. By techniques of our paper we are able to verify that in all of these contexts the space

itself is a PI-space. Thus, while these conditions may encode finer information about the geometry of the space, they do not generate a broader class of spaces than PI-spaces.

The final chapter of this thesis will study differentiation in a general metric space. It is based on the remarkable observation of Cheeger in [28] that general PI-spaces admit a generalization of Rademacher's theorem. Rademacher's theorem states that every Lipschitz function $f \colon \mathbb{R}^n \to \mathbb{R}$ is differentiable almost everywhere. In a metric space, one makes sense of this though the use of chart functions. It is striking that such a result can be generalized to the broad a context of all *PI-spaces*.

Motivated by Cheeger's theorem, a class of spaces satisfying Rademachers theorem in the abstract, called *differentiability spaces*, were introduced. Since his theorem was the only source of examples of such spaces, many authors asked if his assumptions were in some sense necessary. To our knowledge, such questions have been asked my many, and appear in print in at least [12, 38, 72]. We will not attempt to list all of the individuals who have enunciated this question.

Question 3   What are the local geometric conditions on a metric measure space to allow for differentiating Lipschitz functions? Do differentiability spaces possess Poincaré-inequalities in some form?

Related to this question, we study when a metric measure space can be covered by subsets of PI-spaces. Such spaces are called *PI-rectifiable*. In order to answer this question, we needed to study the question of when a given metric measure space can be enlarged to a PI-space.

In a sense, this final question was the starting point of much of the research

in this thesis. Its resolution involved developing the tools to answer the second question, and finding new ways of enlarging a space to a space with a Poincaré inequality. It culminates in Theorem 1.2.62 below, and its auxiliary Theorem 1.2.63, which fully classify PI-rectifiable spaces in terms of differentiability.

Recently, there has been much activity related to differentiability spaces. A very recent development shows that the answer to the previous question involves some subtleties on the target of such Lipschitz functions. This involves combining the results of this thesis with some recent constructions of Andrea Schioppa [6]. As a consequence, we understand better the geometric implications of being a differentiability space, and are able to pose better questions for future study.

The results of this thesis are mostly contained in two submitted publications by the author [52, 53]. While we have restructured the presentation, large portions of the text are copied from these papers. The material in chapter 2 is largely from [52], and the content of chapters 3 and 4 is from [53].

1.2 Notation and Statements

1.2.1 General terminology

Let $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. Denote by $\mathbb{R}_+$ the set of positive real numbers.

We denote metric spaces by (X, d_X) and metric measure spaces (X, d_X, μ_X), when μ_X is a Radon measure on X. Sub-scripts will often be dropped where evident from context. Denote by $B(x, r) = \{y \in X | d(x, y) < r\}$. Especially in the second chapter we will often denote $B(x, r) = B_x(r)$. Unless otherwise stated, throughout this thesis X will be complete and proper, i.e. $B(x, r)$ is pre-compact

for every $x \in X$ and $r > 0$. Further the measure μ will always be assumed to be a Radon measure. This simplifies many limiting arguments and statements of theorems and includes most interesting examples.

Definition 1.2.1. A map $f \colon X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y) is called distance preserving if for all $x, y \in X$

$$d_Y(f(x), f(y)) = d_X(x, y).$$

Note, that there is some variations on terminology. We will define a related notion of *isometry* for Riemannian manifolds below in 1.2.17. These two concepts are easily mistaken for each other.

Definition 1.2.2. Let $L > 0$ be a constant. A map $f \colon X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y) is called $(L-)$Lipschitz if for all $x, y \in X$

$$d_Y(f(x), f(y)) \leq L d_X(x, y). \tag{1.2.3}$$

A map is called $(L-)$bi-Lipschitz if for all $x, y \in X$

$$\frac{1}{L} d_X(x, y) \leq d_Y(f(x), f(y)) \leq L d_X(x, y). \tag{1.2.4}$$

The smallest L such that f is L-Lipschitz is denoted LIP f, and the smallest L such that f is denoted biLIP f. For any constant L such that f is L-Lipschitz, we say that f has distortion L. The constant biLIP f is called *the distortion* of f. We further define local versions of the Lipschitz constant.

$$\text{Lip } f(x) = \limsup_{r \to 0} \sup_{y \in B(x,r)} \frac{|f(x) - f(y)|}{r} \tag{1.2.5}$$

$$\text{lip } f(x) = \liminf_{r \to 0} \sup_{y \in B(x,r)} \frac{|f(x) - f(y)|}{r} \qquad (1.2.6)$$

We will usually assume that Lipschitz functions are defined on the entire space X. Sometimes, functions may *a priori* be defined on a subset. In such circumstances we will use the following extension theorem. Recall, that for a function $f \colon X \to \mathbb{R}$, its restriction to a subset $A \subset X$ is denoted by $f|_A$.

Theorem 1.2.7. *(McShane-Whitney, [71]) Let X be a metric space and $A \subset X$. Then any L-Lipschitz function $f \colon A \to \mathbb{R}^n$ has a $\sqrt{n}L$-Lipschitz extension $\tilde{f} \colon X \to \mathbb{R}^n$ s.t $\tilde{f}|_A = f$.*

Remark: McShane is a good extension result to use because of its generality. However, for manifolds with a lower bound for the sectional curvature bounded by $K \geq \kappa$ better constants could be attained by the use of a generalized Kirzbraun's theorem [95].

For a metric space X we call $N \subset X$ a ϵ-*net* for X if for every $x \in X$ there is a $n \in N$ such that $d(x, n) \leq \epsilon$, and if $x, y \in N$ then $d(x, y) > \epsilon$. Such nets can always be constructed using Zorn's lemma. For a subset $A \subset X$ and $r > 0$, we denote by

$$N_r(A) = \{x \in X | d(x, A) < r\}.$$

Recall, $d(x, A) = \inf_{y \in A} d(x, y)$.

Definition 1.2.8. Let X, Y be metric spaces and $s > 0$. A mapping $f \colon X \to Y$ is called a $(s\text{-})$quasi-isometry if for all $x, y \in X$

$$d_X(x, y) - s \leq d_Y(f(x), f(y)) \leq d_X(x, y) + s,$$

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and $Y \subset N_s(f(X))$.

Using quasi-isometries one can define a distance between compact metric spaces and an associated notion of convergence. For more detailed discussion on convergence of metric and metric measure spaces, we refer the reader to [134]. See also the excellent references [23, 66].

Definition 1.2.9. Let X, Y be compact metric spaces. We denote by $d_{GH}(X, Y)$ the infimum of numbers $s > 0$ such that there are maps $f \colon X \to Y$ and $g \colon Y \to X$ which are s-quasi-isometries. Further, we say that a sequence of compact metric spaces X_i Gromov-Hausdorff converges to another metric space X if

$$\lim_{i \to \infty} d_{GH}(X_i, X) = 0.$$

We also define a measured analog of this definition.

Definition 1.2.10. We say that a sequence of compact metric measure spaces (X_i, d_i, μ_i) measured Gromov-Hausdorff converges to another metric measure space (X, d, μ) if $\lim_{i \to \infty} d_{GH}(X_i, X) = 0$, and there are ϵ_i-quasi-isometries $\phi_i \colon X_i \to X$ such that $\phi_i^*(\mu_i)$ converges weakly to μ.

Here, if $f \colon X \to Y$ is a Borel-measurable map between two metric measure spaces, then $f^*(\mu)(A) = \mu(f^{-1}(A))$ defines the push-forward measure on Y of a measure μ on X.

Recall, that we say that measures $\mu_i \to \mu$ weakly on a proper metric space (X, d), if for every compactly supported continuous function $f \colon X \to \mathbb{R}$ we have

$$\lim_{i \to \infty} \int f \, d\mu_i = \int f \, d\mu.$$

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We can also define a notion of pointed convergence for proper metric measure spaces.

Definition 1.2.11. A sequence of proper pointed metric (measure) spaces (X_i, x_i) Gromov-Hausdorff converges to another metric measure space (X, x) if $\overline{B(x_i, R)}$ converges to $\overline{B(x, R)}$ for every $R \geq 0$ in the (measured) Gromov-Hausdorff sense. The balls are equipped with the restricted metric (and measure).

It is often useful and necessary to impose a type of finite-dimensionality condition on a metric space. This property ensures, that the size of ϵ-nets can be bounded.

Definition 1.2.12. Let $D > 0$ be a constant. A metric space (X, d) is called D-metric doubling if for every $x \in X$ and $r > 0$, there are points $x_1, \ldots, x_D$ such that

$$B(x, r) \subset \bigcup_{i=1}^{D} B(x_i, r/2).$$

A metric measure space has a finer variant of doubling, called measure doubling.

Definition 1.2.13. Let $D > 0$ be a constant. A proper metric measure space (X, d) equipped with a Radon measure is called D-measure doubling if for every $x \in X$ and $\infty > r > 0$:

$$\frac{\mu(B(x, 2r))}{\mu(B(x, r))} \leq D. \tag{1.2.14}$$

We say that the space is locally $((D, r_0)-)$doubling if there exists a $r_0 > 0$ such that for any $x \in X$ the previous inequality holds for $r < r_0$.

Given a subset $A \subset X$ of a metric space, its diameter is denoted by

$$\text{Diam } (A) = \sup_{a,b \in A} d(a,b). \tag{1.2.15}$$

If X is a proper metric space, with distance function d, we denote for $p \in X$ by $T_p(X)$ the set of tangents of X, i.e. the pointed Gromov-Hausdorff-limits of $(X, p, d/r_i)$ for $r_i \to \infty$ (when such limits exist). For metric measure spaces $T_p(X)$ denotes all the measured limits of $(X, p, \mu/\mu(B(p, 1/r_i)), d/r_i)$ for any sequence $r_i \to \infty$. For doubling metric (measure) spaces tangents exist in abundance due to Gromov's compactness theorem. We state it in the following form, but for more discussion we refer the reader to [66]. To state it, we denote by $N(x, \epsilon, r)$ the size of the smallest ϵ-net on $B(x, r) \subset X$. We have by compactness that $N(x, \epsilon, r) < \infty$ for proper metric spaces, and $N(x, \epsilon, r) \leq D^{\log_2(r/\epsilon)+2}$ for a D-doubling metric space.

Theorem 1.2.16. *(Gromov's compactness theorem) Assume (X_i, d_i, p_i) is any sequence of pointed proper metric spaces such that for every $r > 0$ there is a $N_r \colon \mathbb{R}_+ \to \mathbb{N}$ with the property that*

$$N(p_i, \epsilon, r) \leq N_r(\epsilon).$$

Then a sub-sequence of (X_i, d_i, p_i) converges to a proper metric space. In particular, if (X_i, d_i, p_i) are $D-$metric doubling for a uniform $D > 0$, then there is a convergent sub-sequence of such spaces.

If (X_i, d_i, μ_i, p_i) is a pointed sequence of proper $D-$measure doubling spaces for a uniform $D > 0$, then a subsequence measured Gromov-Hausdorff converges.

Given a subset $A \subset X$, where (X, d) is a metric space, the function $d_A = d|_{A \times A}$ defines the restricted metric on A. Further, if we have a metric measure space

(X, d, μ), we define the restricted measure as $\mu_A(E) = \mu(E)$ for $E \subset A$. In this definition, we will assume that A is a Borel subset and μ is a Borel measure.

1.2.2   Lipschitz maps and embeddings

A map $f \colon X \to \mathbb{R}^n$ is an embedding if it is a homeomorphism onto its range. A bi-Lipschitz map $f \colon X \to \mathbb{R}^n$ is refered to as a *bi-Lipschitz embedding*. Such a map preserves distances up to a factor. This is slightly ambiguous, since in literature one can also consider bi-Lipschitz embeddings with non-Euclidean targets. However, in this thesis we restrict attention to this context. These mappings are interesting since they preserve such notions as rectifiable curves and Hausdorff-measure (see [104]). With this notion at hand, we make the general question of the introduction precise.

Embedding Question   When does a complete Riemannian manifold M, or its subset, possess a bi-Lipschitz embedding? How can the distortion and target dimension of such embeddings be controlled by the parameters describing the manifold.

This is an old question, and for some general references the reader can consult [70, 71]. It turns out, that this problem is quite hard in general. In fact, as discussed in [131], it is hard to come up with complete theorems in this direction. It seems more reasonable to come up with theorems that address the existence of bi-Lipschitz embeddings for classes of manifolds. Motivated by a question of Pete Storm and Bruce Kleiner, we are interested in understanding this question when X is a manifold, orbifold or a subset thereof. Denote a n-dimensional Riemannian

manifold by (M^n, g).

Definition 1.2.17. Let $m \geq n$ be integers. A smooth embedding $f \colon M^n \to N^m$ between a n-dimensional Riemannian manifolds (M^n, g_M) and a m dimensional manifold (N^m, g_N) is called an isometry if for all $x \in M^n$

$$df_* g_N = g_M,$$

where df_* denotes the pull-back operator for tensors.

An equivalent, and more intuitive, way of stating this is that for every $x \in M$ and every $\epsilon > 0$, there a $\delta_{x,\epsilon} > 0$ such that $f|_{B(x, \delta_{x,\epsilon})}$ is $(1 + \epsilon)$-bi-Lipschitz. In other words, a smooth embedding is an isometry if and only if it is almost preserves distances infinitesimally. The Nash embedding theorem [111] guarantees that such embeddings exist in great abundance. However, such a map can distort lengths at large scales by arbitrary factors. Note, that the lengths in the target space are measured "as the crow flies". Specifically, our interest is in preserving distances globally, and controlling the distortion uniformly over some parameters describing the spaces.

Example For intuition, consider the family of surfaces

$$S_N = \{x^2 + y^2 + N^2 z^2 = 1\}.$$

Denote by g_N the restricted metric on this two-dimensional surface. This defines boundaries of ellipsoids in $\mathbb{R}^3$ which are isometrically embedded in $\mathbb{R}^3$ but the natural embedding is not bi-Lipschitz. The distance along the surface between the points $x_\pm = \left(0, 0, \pm \frac{1}{N}\right)$ is $d(x_+, x_-) \sim 2$, but in $\mathbb{R}^3$ the straight-line distance

is $|x_+ - x_-| = \frac{2}{N}$. Thus, the isometric embedding of S_N does not have a well-controlled bi-Lipschitz constant. However, if we define the map $L\colon S_N \to \mathbb{R}^3$ by sending $(x, y, z) \to \left(x, y, z + \left(1 - \sqrt{x^2 + y^2}\right) H(z)\right)$. Here $H(z) = 1$ for $z \geq 0$ and otherwise $H(z) = 0$. Such a map can be shown to be C-bi-Lipschitz, but it is clearly not an isometry. This constant C can be chosen independent of N.

Another related notion is that of a bi-Hölder embedding.

Definition 1.2.18. Let $L > 0$ be a constant. A map $f\colon X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y) is called $(\alpha-)$Hölder (with constant L) if for all $x, y \in X$

$$d_Y(f(x), f(y))^\alpha \leq L d_X(x, y)^\alpha. \tag{1.2.19}$$

A map is called $(\alpha-)$bi-Hölder (with constant L) if for all $x, y \in X$

$$\frac{1}{L} d_X(x, y)^\alpha \leq d_Y(f(x), f(y))^\alpha \leq L d_X(x, y)^\alpha. \tag{1.2.20}$$

The classical Assouad embedding theorem states that it is much easier to construct bi-Hölder embeddings. For a proof and related discussion consult [7,70,109].

Theorem 1.2.21. [7] (Assouad embedding theorem) Let (X, d) be a doubling metric space. Then for every $0 < \alpha < 1$ there exists a α-bi-Hölder embedding $f\colon X \to \mathbb{R}^N$, where the distortion and N can be bounded in terms of α and the doubling constant D.

As indicated in the examples of section 1.1, there are cases when α can not be taken to be $\alpha = 1$. More subtle examples of this arise in the work of Laakso related to the quasiconformal Jacobian problem and are mentioned in [93]. These

are based on other similar constructions in [92]. We give some remarks on this connection.

Quasiconformal Jacobian problem: The quasiconformal Jacobian problem asks when a given locally integrable function $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is the Jacobian derivative of some quasiconformal (or, equivalently, quasisymmetric) homeomorphism $f \colon \mathbb{R}^n \to \mathbb{R}^n$. These questions correspond to a problem of bi-Lipschitz paramtrization of a metric space arising via a conformal deformation [46]. Characterizing such functions has proven incredibly hard. In fact, the work of Semmes [46, 129, 131] studies conditions on the weights that would guarantee a weaker statement; that of bi-Lipschitz embeddings. This connects directly to our question on embedding manifolds, when one restricts attention to manifolds (M, g) which are conformal to Euclidean space.

Every subset of Euclidean space $\mathbb{R}^n$ is 2^n-measure doubling and thus 4^n-metric doubling. Further, if $f \colon X \to Y$ is L-bi-Lipschitz and Y is D-doubling, then X is $D^{2\log_2(L)+5}$-doubling. Thus, a pre-requisite for any metric space to be embedded in Euclidean space is that it is doubling. In the manifold context this is ensured by a lower bound on the sectional curvature $K \geq \kappa$.

Example: Graph approximations of surfaces Let X be a compact geodesic metric space. Take a fine ϵ-net N, associate to each $n \in N$ a vertex of a graph and attach edges of length ϵ to any pair of vertices at distance at most 5ϵ from each other. Call such a metric graph G_ϵ. It is possible to show that, for some sub-sequence $\epsilon \to 0$ the sequence G_ϵ converges to a metric space $\overline{X}$, which is C-bi-Lipschitz to X for some universal C. Further any graph can be approximated by a

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two-dimensional surface S by inflating the vertices into small spheres, and gluing tubes corresponding to edges. Thus, any compact geodesic metric space is, up to a bi-Lipschitz deformation, close to a smooth surface. If it would be possible to uniformly embed two-dimensional closed surfaces, it would by a limiting argument follow that X could also be embedded. As noted, examples exist in abundance of doubling metric spaces X that do not embed well or at all.

Our techniques for constructing embeddings need also an upper bound for sectional curvature. Thus we prove the following.

Theorem 1.2.22. *Let $D > 0$. Every bounded subset A with Diam $(A) \leq D$ in a n-dimensional complete Riemannian manifold M^n with sectional curvature $|K| \leq 1$ admits a bi-Lipschitz embedding $f \colon A \to \mathbb{R}^N$ with distortion less than $C(D, n)$ and dimension of the image $N \leq N(D, n)$.*

It turns out that the statement can be generalized for any orbifold as follows. For terminology relating to orbifolds we refer to section **??**.

Theorem 1.2.23. *Let $D > 0$. Every bounded subset A with Diam $(A) \leq D$ in a n-dimensional complete Riemannian orbifold O^n with sectional curvature $|K| \leq 1$ admits a bi-Lipschitz embedding $f \colon A \to \mathbb{R}^N$ with distortion less than $C(D, n)$ and dimension of the image $N \leq N(D, n)$.*

In particular we have the following non-trivial and concrete result. This statement and its proof, combined with an approximation argument, form the core of the proof of the above theorems.

Theorem 1.2.24. *Every complete flat and elliptic orbifold O of dimension n admits a D-bi-Lipschitz mapping $f \colon O \to \mathbb{R}^N$ with $D \leq D(n)$ and dimension of the*

image $N \leq N(n)$. The constants $D(n)$ and $N(n)$ depend only on the dimension.

In particular, for any discrete subgroup $\Gamma \subset \mathrm{Isom}(\mathbb{R}^n)$, the induced Riemannian flat orbifold $O = \mathbb{R}^n/\Gamma$ with its quotient metric has a $D(n)$-bi-Lipschitz embedding into Euclidean space of dimension $N(n)$. The distortion depends only on the dimension, and not on the group. Also, the quotient may be non-compact. Our bounds for the distortion and dimension are of the order $D(n) = O(e^{Cn^4 \ln(n)})$ and $N(n) = O(e^{Cn^2 \ln(n)})$. These bounds are likely highly non-optimal, but show that the distortion is independent of the group.

Example: Klein bottle. We wish to illustrate the non-triviality of the above theorem for even simple cases. Consider the "collapsed" Klein bottle K_ϵ. This can be given as $K_\epsilon = \mathbb{R}^2/\Gamma_\epsilon$, where Γ_ϵ is the group of isometries generated by the maps $R_\epsilon\colon (x,y) \to (-x, y+\epsilon)$, and $T\colon (x,y) \to (x+1, y)$. It is hard to visualize, how K_ϵ could be represented in Euclidean space in a bi-Lipschitz way. Yet the above theorems provide such an embedding with a bound on distortion independent of ϵ. The space K_ϵ is in fact a manifold.

Example: Torus. A flat torus is isometric to $T = \mathbb{R}^n/\Lambda$, where $\Lambda \subset \mathbb{R}^n$ is a discrete and co-compact group acting by translations. Theorems 1.2.22 and 1.2.23 state that it is possible to embed T into $\mathbb{R}^N$ in a bi-Lipschitz way, where N and the distortion only depend on the dimension n. In particular, the lattice may be very complicated. In general, T may have up to n "collapsed directions", and very small volume. The distortion of such examples was first bounded by Naor and Khot in [87], and later in a more optimal way by Haviv and Regev [69]

Example: Group quotient. A further complication arises from the following example. Consider any finite group G with a faithful representation as isometries of $\mathbb{R}^n$. Recall, any faithful representation may be converted to one by isometries. Consider the associated Riemannian orbifold $O = \mathbb{R}^n/G$. Again, Theorem 1.2.24 guarantees a bi-Lipschitz embedding for O with a bound for the distortion only depending on n. The reader is instructed to consider the case where $G = S_n$, the permutation group on n symbols, and acting on $\mathbb{R}^n$ by permutation of co-ordinates. The group quotient $\mathbb{R}^n/S_n$ plays a crucial role in related work by Naor and Neiman on "Snowflake universality" [5].

1.2.3   Poincaré-inequalities

Let be a proper metric measure space (X, d, μ) with a Radon measure μ and f a measurable function. We say that $f \in L^1_{loc}$, if for every $B(x, r) \subset X$ we have

$$\int_{B(x,r)} |f| \, d\mu < \infty.$$

Such a function is called locally integrable. For any bounded positive-measure set $A \subset B(x, R) \subset X$ we define

$$f_A = \fint_A f \, d\mu = \frac{1}{\mu(A)} \int_A f \, d\mu.$$

We will call a map $\gamma \colon I \to X$ from a compact interval I a *curve*. The length is defined in terms of the total variation

$$\text{len}(\gamma) = \sup_{t_0 < t_1 < \cdots < t_n \in I, t_i \in I} \sum_{i=1}^n d(\gamma(t_i), \gamma(t_{i-1})).$$

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A curve is called rectifiable if $\text{len}(\gamma) < \infty$. For a rectifiable curve it is always possible to find a parametrization by length. A curve $\gamma \colon I \to X$ is said to be parametrized by length if $\text{len}(\gamma|_{[a,b]}) = |b - a|$ for every $[a, b] \subset I$. See [144] for more details.

While derivatives, a priori, do not make sense in a metric setting, it is relatively easy to define a notion of "absolute value of a gradient". This notion was introduced by Heinonen and Koskela in [74]. If (X, d) is a metric space and f is a bounded Borel-function on X, we call a measurable non-negative Borel-function $g \colon X \to \mathbb{R}_+ \cup \{\infty\}$ an upper gradient of f if for every rectifiable curve $\gamma \colon [a, b] \to X$ in X we have

$$|f(\gamma(1)) - f(\gamma(0))| \leq \int_a^b g(\gamma(t)) \, ds. \tag{1.2.25}$$

Here, ds represents integration with respect to the total variation measure of γ. In fact, if γ is parametrized by length we have $ds = dt$.

Examples: Consider the space $\mathbb{R}^n$ with its usual metric and f a C^1-function. Then $g = |\nabla f|$ is an upper gradient. In fact, it is not hard to show, that if g is any upper gradient then $g \geq |\nabla f|$ almost everywhere. If f is a Lipschitz function, then by Rademachers theorem (see 1.2.50) it is almost everywhere differentiable. In such a case we can define $g = |\nabla f|$, where defined, and $g = \infty$ otherwise. Such a function is Borel-measurable and by a direct argument an upper gradient. For general metric spaces (X, d) the functions lip f and Lip f are both upper gradients for a Lipschitz function f.

Poincaré-inequalities are a way of relating infinitesimal oscillations of func-

tions to their global oscillations. There are slightly different versions of Poincaré-inequalities, but we choose the following in light of its simplicity.

Definition 1.2.26. Let $p \in [1, \infty)$, $C, C' > 0$ be constants. We say that a proper metric measure space (X, d, μ) equipped with a Radon measure satisfies a $(1, p)$-Poincaré-inequality with constants (C, C') if for every $r > 0$, every $x \in X$ and every Lipschitz function $f \colon X \to \mathbb{R}^n$.

$$\fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq Cr \left(\fint_{B(x,C'r)} \mathrm{lip} \ f^p \, d\mu \right)^{\frac{1}{p}}. \tag{1.2.27}$$

Additionally, we say that the metric measure space (X, d, μ) satisfies a local $((1, p, r_0)$–)Poincaré-inequality if for some $r_0 > 0$ the previous holds for all $r < r_0$. A space satisfies a local $(1, p)$-Poincaré-inequality if the aforementioned holds for some $r_0 > 0$.

There are several equivalent definitions for Poincaré-inequalities. We will discuss these in more detail in chapter 3. Note, that our spaces will be assumed locally complete. This avoids some well known counter-exampels, such as [90].

A measure-doubling space satisfying a $(1, p)$-Poincaré-inequality is referred to as a p-Poincaré space (following [72]) or a PI-space. The latter terminology is due to Kleiner and Cheeger [38].

Definition 1.2.28. A metric measure space is called a PI-space if there exist constants $p \geq 1$ and $r_0, C, C' > 0$ such that it satisfies a local $(1, p, r_0)$-Poincaré-inequality with constants (C, C').

Originally, PI-spaces arose as a natural setting to study quasiconformal and quasisymmetric mappings. Most importantly, the classical theory of these map-

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pings, which originally was developed in the Euclidean context, could be generalized to a large extent to this class of spaces. A foundational paper leading this approach was [74]. The results an techniques used there proved later to connect with questions in geometric group theory [17] (see also the survey [88]). Due to the central role of these spaces, it became an important problem to better understand which geometric properties characterize PI-spaces.

PI-spaces arise in both classical contexts and some more unusual spaces. We recall here a few of the most well known examples. These include the two examples considered in section 1.1.

Example: Ricci-bounded manifolds Let (M, g) be a complete n-dimensional Riemannian manifold with a Ricci-curvature bound $\text{Ric} \geq -K(n-1)$ for some $K > 0$. Then (M, g) satisfies a local $(1, p, r_0)$-Poincaré-inequality for every r_0. This can be derived from the proof in [29].

Example: Carnot group Let (G, g) be a so-called Carnot group. It is defined as a Lie group endowed with sub-spaces of the Lie algebra $L_i \subset \mathfrak{g}$ for $i = 1, \ldots, k$ such that $[L_1, L_j] = L_{1+j}$, and such that $L_i \cap L_j = \{0\}$ for $i < j$ and $\bigoplus L_i = \mathfrak{g}$. Each L_i induces via left-translations a sub-bundle of the tangent-bundle of G, which we identify with itself. Corresponding to such a Lie-group and any fixed Riemannian norm $|| \cdot ||$ on L_1, we can define a sub-Riemannian metric by

$$d(x, y) = \inf \int_0^1 ||\gamma'(t)||dt,$$

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where $\gamma\colon [0,1] \to G$ is a Lipschitz curve with $\gamma'(t) \in L_1$ for almost every t. Let the measure μ be the corresponding Haar measure on G. By results from [78] such a space satisfies a Poincaré-inequality, and further by results in [107] one can show that the space is doubling.

Example: Gromov-Hausdorff limits of PI-spaces   Let (X_i, d_i, μ_i, x_i) be a sequence of pointed proper PI-spaces that satisfy a local $(1, p, r_0)$-Poincaré-inequality with constants (C, C') for some fixed r_0, C, C', p, and such that they are all D-measure doubling for a fixed D. Then there exists a sub-sequence such that $(X_i, d_i, \mu_i/\mu_i(B(x_i, 1)), x_i)$ converges to a PI-space. See the arguments in [28, 84].

Well known characterizations of PI-spaces appeared in [74] and [84]. However, difficulties arise when trying to apply these in practice. Thus, we would like to give new tools to prove $(1, p)$-Poincaré-inequalities using *a priori* weaker conditions. Originally, we were motivated by the goal of describing the local geometry of differentiability spaces (see subsection 1.2.4). However, we have noticed that our condition arises naturally in many applications.

The main improvement compared to the prior work in [74, 84] is that we can *a priori* work with curve fragments instead of curves. By a curve-fragment we mean a Lipschitz map $\gamma\colon K \to X$ from a compact set $K \subset \mathbb{R}$. The length of such a curve fragment is defined as

$$\operatorname{len}(\gamma) = \sup_{x_i} \sum_{i=1}^{n} d(\gamma(x_{i+1}), \gamma(x_i)), \tag{1.2.29}$$

where $x_i \in K$ and $x_i \leq x_j$ for $i \leq j$. Note that this is formally the same as for

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curves, except taking into account the change in the domain.

We say that a curve fragment connects two points $x, y \in X$ if $\gamma(\min(K)) = x, \gamma(\max(K)) = y$. Usually, we will translate the domain in such a way that $\min(K) = 0$. Also, often we will dilate the domain so that γ is 1-Lipschitz. In this case we have $\mathrm{len}(\gamma) \leq \max(K)$, which is often the most useful estimate. Associated to a curve fragment we can defined the undefined portion.

$$\mathrm{Undef}(\gamma) = [\min(K), \max(K)] \setminus K. \tag{1.2.30}$$

This set can also be called the set of gaps. Often, it will be useful to decompose $\mathrm{Undef}(\gamma)$ into maximal disjoint open intervals (a_i, b_i), indexed by natural numbers i such that

$$\mathrm{Undef}(\gamma) \subset \bigcup_i (a_i, b_i). \tag{1.2.31}$$

Curve fragments arise naturally in the study of differentiability spaces. Their importance was first observed by David Bate in [11] (see also [126]).

Examples: Consider $x, y \in X$. The pair of points can always be connected by a curve fragment as follows. Define $K = \{0, d(x,y)\}$, and $\gamma\colon K \to X$ as $\gamma(0) = x, \gamma(d(x,y)) = y$. This is a 1-Lipschitz curve fragment with $\mathrm{len}(\gamma) = d(x,y)$ and $\mathrm{Undef}(\gamma) = (0, d(x,y))$. We are interested in curve fragments, where $|\mathrm{Undef}(\gamma)| \leq \delta d(x,y)$ for some $0 < \delta < 1$.

Consider another example. Take a totally disconnected compact subset $E \subset [0,1]$ with $\mu(E) > 0$. Then $x, y \in E$ can be connected as follows. Let $K = [x,y] \cap E$, and $\gamma\colon K \to [0,1]$ is the inclusion map. Then $\mathrm{len}(\gamma) = |x - y|$, and $\mathrm{Undef}(\gamma) =$

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$[x, y] \setminus E$. We see, that the set K where γ is defined can be totally disconnected. At a density point of E, this set can be made to have nearly full measure in the interval $[x, y]$. This allows us to have a notion of "well-connectedness" by curve fragments even for sets that may be totally disconnected.

In terms of this notion of curve fragments we present our quantitative connectivity condition. Recall, that for any $n \in \mathbb{N}$ and any subsets $A \subset \mathbb{R}^n$ we will denote by $|A|$ the Lebesgue measure of A.

Definition 1.2.32. Let[1] $0 < \delta$, $0 < \epsilon < 1$ and $C > 1$ be given. If $(x, y) \in X \times X$ is a pair of points with $d(x, y) = r > 0$, then we say that the pair (x, y) is (C, δ, ϵ)-connected if for every Borel set E such that $\mu(E \cap B(x, Cr)) < \epsilon\mu(B(x, Cr))$ there exists a 1-Lipschitz[2] curve fragment $\gamma \colon K \to X$ connecting x and y, such that the following hold.

1. $\min(K) = 0$

2. $\operatorname{len}(\gamma) \leq Cd(x, y)$

3. $|\operatorname{Undef}(\gamma)| < \delta d(x, y)$

4. $\gamma^{-1}(E) \subset \{0, \max(K)\}$

We call (X, d, μ) a $(C, \delta, \epsilon, r_0)$-connected space, if every pair of points $(x, y) \in X$ with $r = d(x, y) \leq r_0$ is (C, δ, ϵ)-connected.

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[1] The definition is only interesting for $\delta < 1$. However, we allow it to be larger to simplify some arguments below. If $\delta \geq 1$ then the curve fragment could consist of two points $\{0, d(x, y)\}$, and $\gamma(0) = x, \gamma(d(x, y)) = y$.

[2] According to Lemma 3.2.7 we could simply assume some Lipschitz bound, but it is easier to normalize the situation.

We say that X is uniformly $(C, \delta, \epsilon, r_0)$-connected along S, if every $(x, y) \in X \times X$ with $x \in S$ and with $d(x, y) \le r_0$ is (C, δ, ϵ)-connected in X.

If (X, d, μ) is $(C, \delta, \epsilon, r_0)$-connected for all r_0, we simply say that (X, d, μ) is (C, δ, ϵ)-connected.

The set E will be referred to as an obstacle, and we will say that γ avoids the obstacle E. Thus, the property could also be called a quantitative avoidance property. In the definition, we are forced to allow γ to intersect E at the end points, because x and y may lie in E.

The set E can also be thought of as locally separating x and y. This is analogous to the role of a separating hypersurface S in Cheeger's lower bound for eigenvalues [27]. The class of sets E considered in the above definition is somewhat inconsequential. Since we are interested in Radon measures on proper metric spaces, μ is always inner and outer regular (similar arguments appear in [104]), and thus E could be assumed to be compact or open. This is useful for some arguments.

We note, that the above definition is formally motivated by that of Muckenhoupt weights. For purposes of illustrating this comparison, we define the class of Muckenhoupt weights A_∞. We state the definition for any metric measure space, while the reader is instructed to focus on the case of $\mathbb{R}^n$. Also, the class of Muckenhoupt weights on $\mathbb{R}^n$ can be defined in several different ways, and some of these definitions are inequivalent on metric measure spaces. We do not delve here into these details. One can consult [81] for some related discussion.

Definition 1.2.33. Let (X, d, μ) be a metric measure space. We say that $\nu \in A_\infty(\mu)$, if ν is absolutely continuous with respect to μ and the following is true. There are constants $0 < \delta, \epsilon < 1$ such that whenever $E \subset B(x, r) \subset X$, then

$$\nu(E) \le \delta\nu(B(x,r)) \implies \mu(E) \le \epsilon\mu(B(x,r)). \tag{1.2.34}$$

If we replace $\mu(E)$ by $\mathrm{Undef}(\gamma)$, then the definitions are roughly the same. The set E in both cases can be thought of as an obstacle.

Remark: If we were to investigate our definition in more detail, we would observe that the analogy of A_p-weights is slightly misleading. The more close analogy is to so called $A_{p,1}$-weights discussed in [91, 96]. While the A_p-condition characterized boundedness of the Hardy-Littlewood maximal function M from L^p to L^p, the $A_{p,1}$-condition characterizes boundedness from $L^p \to L^{p,\infty}$. Here, $L^{p,\infty}$ is the Lorentz space of measurable functions f with $\sup_{t>0} t^p \mu(|f| > t) < \infty$. It is known that $A_p \subset A_{p,1}$ strictly. Further, the $A_{p,1}$-condition does not improve to $A_{q,1}$ for any $q < p$. Similarly, our condition does not self-improve in this fashion.

If a space is quasiconvex, then in the previous definition we could replace curve fragments by curves. Let $L \ge 1$ be a constant. A space is called $(L-)$quasiconvex, if for every pair of points $x, y \in X$ there exists a rectifiable curve $\gamma \colon I \to X$ with $\gamma(\min(I)) = x, \gamma(\max(I)) = y$ where $I \subset \mathbb{R}$ is a compact interval and $\mathrm{len}(\gamma) \le L d(x,y)$. A space is locally quasiconvex, if the aforementioned holds when $d(x,y) \le r_0$ for some fixed $r_0 > 0$. Every PI-space is locally quasiconvex, see [28]. For quasiconvex spaces the following condition is equivalent to Definition 1.2.32.

Definition 1.2.35. Let $0 < \delta$, $0 < \epsilon < 1$ and $C > 1$ be given. If $(x,y) \in X \times X$ is a pair of points with $d(x,y) = r > 0$, then we say that the pair (x,y) is (C, δ, ϵ)-connected by a curve if for every Borel set E such that $\mu(E \cap B(x, Cr)) < \epsilon\mu(B(x, Cr))$ there exists a 1-Lipschitz curve $\gamma \colon I \to X$ connecting x and y, such

that the following hold.

1. $\text{len}(\gamma) \leq C d(x, y)$

2. $|\gamma^{-1}(E)| < \delta d(x, y)$

Similarly we say that (X, d, μ) is $(C, \delta, \epsilon, r_0)$-connected by curves, if every pair of points $(x, y) \in X$ with $r = d(x, y) \leq r_0$ is (C, δ, ϵ)-connected by a curve.

If (X, d, μ) is $(C, \delta, \epsilon, r_0)$-connected by curves for all r_0, we simply say that (X, d, μ) is (C, δ, ϵ)-connected by curves.

The equivalence is stated in Lemma 3.3.10. Note that every locally $(C, \delta, \epsilon, r_0)$-connected space is also quasiconvex by an argument in Theorem 3.3.4. However, the reason we need also Definition 1.2.32, is to introduce the notion of (C, δ, ϵ)-connected *along* a subset $S \subset X$. In this case, it makes a difference, which definition we use. The notion of connectivity along a subset is somewhat like a relative version of the definition. It arises in the study of differentiability spaces and plays a crucial role in chapter 4. It will be discussed briefly in section 1.2.4.

Example of $\mathbb{R}^n$: Take $\mathbb{R}^n$ with the Lebesgue measure λ and usual metric. It is straightforward to show using polar-co-ordinates (or modulus bounds), that there is a constant $C > 0$ such that $\mathbb{R}^n$ is $(2, C\epsilon^{1/n}, \epsilon)$-connected for every $\epsilon > 0$. Also, rather trivially, $\mathbb{R}$ is $(1, 2\epsilon, \epsilon)$-connected.

Example of Muckenhoupt weights: Take $\mathbb{R}^n$ with the usual metric and the measure $\mu = w\lambda$, where $w \in A_p(\lambda)$ and λ is Lebesgue measure. The class A_p consists of those locally integrable functions which are Muckenhoupt-weights, i.e.

$$\left(\fint_B w \, d\lambda \right) \left(\fint_B w^{1-p} \, d\lambda \right)^{\frac{1}{p-1}} \leq C$$

for some $C < \infty$ and all balls $B = B(x,r) \subset \mathbb{R}^n$. These weights have many equivalent characterizations and they arise naturally in the study of weighted bounds for maximal functions and singular integrals. One of the curious properties of these weights are the following characterization (compare Definition 1.2.33).

$$A_\infty(\lambda) = \bigcup_{1 \leq p < \infty} A_p.$$

We will discuss these weights a little more in section 3.4. Also, see [135] and the original paper by Muckenhoupt [106]. An immediate consequence of the definition and Hölders-inequality is that

$$
\begin{aligned}
\fint_B f \, d\lambda &\leq \left(\fint_B f^p w \, d\lambda \right)^{\frac{1}{p}} \left(\fint_B w^{1-p} \, d\lambda \right)^{\frac{p}{p-1}} \\
&\leq C^{\frac{1}{p}} \left(\frac{1}{\mu(B)} \int_B f^p w \, d\lambda \right)^{\frac{1}{p}}.
\end{aligned}
\tag{1.2.36}
$$

Applying this to $f = 1_E$ gives the following.

$$\frac{|E|}{|B|} \leq C^{\frac{1}{p}} \left(\frac{\mu(E)}{\mu(B)} \right)^{\frac{1}{p}}. \tag{1.2.37}$$

Thus, it is not hard to see that given such a μ, the space $(\mathbb{R}^n, | \cdot |, \mu)$ is $(2, C'\epsilon^{1/(np)}, \epsilon)$-connected for every $\epsilon > 0$.

Our main theorem in terms of this notion is as follows.

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Theorem 1.2.38. *A (D, r_0)-doubling complete metric measure space (X, d, μ) admits a local $(1, p)$-Poincaré-inequality for some $1 \leq p < \infty$ if and only if it is locally (C, δ, ϵ)-connected for some $0 < \delta, \epsilon < 1$. Both directions of the theorem are quantitative in the respective parameters.*

Remark: By Lemma 3.3.2 (C, δ, ϵ)-connectivity itself already implies doubling.

As noted before, a PI-space is always locally L-quasiconvex. However, *a priori* we only have curve fragments connecting pairs of points. Thus, we need a way of making the gaps smaller. We are thus lead to the following definition, where the size of the gaps is more effectively controlled by the size of the obstacle E. The analogous statement for Muckenhoupt weights is that the condition 1.2.34 (on $\mathbb{R}^n$) immediately improves to 1.2.37. See also the discussion in section 3.4.

Definition 1.2.39. We call a metric measure space (X, d, μ) finely α-connected with parameters (C_1, C_2) if for any $0 < \tau < 1$ the space is $(C_1, C_2 \tau^\alpha, \tau)$-connected. Further, we say that the space is locally finely (α, r_0)-connected with parameters (C_1, C_2) if it is $(C_1, C_2 \tau^\alpha, \tau, r_0)$-connected for any $0 < \tau < 1$. If we do not explicate the dependence on the constants, we simply say (X, d, μ) is finely α-connected if constants C_1, C_2 exist with the previous properties. Similarly we define locally finely (α, r_0)-connected spaces.

For example, note that $\mathbb{R}^n$ is finely $\frac{1}{n}$-connected. Similarly, the space arising from gluing two copies of $\mathbb{R}^n$ along the origin is also $\frac{1}{n}$-connected. If the Lebesgue measure λ on $\mathbb{R}^n$ is deformed by an A_p-Muckenhoupt weight w, then the space $(\mathbb{R}^n, |\cdot|, wd\lambda)$ is $\frac{1}{pn}$-connected. This follows from the examples above.

As noted above, any A_∞-weight on $\mathbb{R}^n$ belongs to some class A_p for $1 \leq p < \infty$.

The following statement is an analog of this result for our notion of connectivity. A close examination of the proof in [135, Section V] shows that this result can be proven by similar means.

Theorem 1.2.40. *Assume $0 < \delta < 1$ and $0 < \epsilon, r_0, D$. If (X, d, μ) is a $(C, \delta, \epsilon, r_0)$-connected metric measure space, then there exist $0 < \alpha, C_1, C_2$ such that it is also finely $(\alpha, r_0/(8C_1))$-connected with parameters (C_1, C_2). We can choose $\alpha = \frac{\ln(\delta)}{\ln(\frac{\epsilon}{2M})}$, $C_1 = \frac{C}{1-\delta}$ and $C_2 = \frac{2M}{\epsilon}$, where $M = 2\max(D^{-\log_2(1-\delta)+1}, D^3)$. Here D is the implied doubling constant from 3.3.2.*

The previous theorem is the first step in proving Theorem 1.2.38. The final step consists of a more quantitative version of this statement, which is the following.

Theorem 1.2.41. *Let (X, d, μ) be a locally finely (α, r_0)-connected metric measure space, then for any $p > \frac{1}{\alpha}$ there exists C_{PI}, C_1 such that the space has a local $(1, p, r_0/(8C_1), C_{PI}, 2C_1)$-Poincaré-inequality. In short the space is a PI-space.*

We can set $M = 2(D^4)^{\frac{1}{p\alpha - 1}}$, $\delta = \left(\frac{D^4}{M^p}\right)^{\alpha}$, $C_1 = \frac{C}{1-\delta}$, $C_3 = \frac{C_1 M}{1 - M\delta}$ and $C_{PI} = 2C_3$. Here D is an implied Doubling constant from 3.3.2.

Next, we present some applications of these theorems and concepts. The first result concerns metric spaces with weak Ricci bounds. These spaces originally arose following the work of Cheeger and Colding on Ricci limits [29, 30]. Many different definitions appeared such as different definitions for $CD(K, n)$ [97, 136, 137], $CD^*(K, n)$ [10] and a strengthening $RCD(K, n)$ [2, 4]. Ohta also defined a very weak form of a Ricci bound by the *measure contraction property* and introduced $MCP(K, N)$-spaces [113]. $MCP(K, n)$ spaces are interesting partly because they are known to include some very non-Euclidean geometries such as the Heisenberg group and certain Carnot-groups [79, 121].

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Spaces satisfying one of the stronger conditions ($CD(K,N)$, $RCD(K,N)$ or $RCD(K,N)^*$) were all known to admit Poincaré-inequalities [118]. Further it was known that a $MCP(K,N)$-space which was non-branching also would admit a $(1,1)$-Poincaré-inequality. This was observed by Renesse [146], whose proof was essentially a repetition of classical arguments in [29]. However, Rajala had conjectured that this assumption of non-branching was inessential. We are able to prove the following.

Theorem 1.2.42. *If (X,d,μ) is a $MCP(K,n)$ space, then it satisfies a local $(1,p)$-Poincaré-inequality for $p > n+1$. Further if $K \geq 0$ then it satisfies a global $(1,p)$-Poincaré-inequality.*

We are left with the following open problem.

Open Question: Does every $MCP(K,n)$-space admit a local $(1,1)$-Poincaré-inequality?

We next consider some more general self-improvement phenomena for Poincaré-inequalities. Originally self-improvement results arose in the context of reverse Hölder-inequalities and A_p-weights [39, 62, 77, 106, 135]. In an important paper Keith and Zhong proved that in a doubling complete metric measure space a $(1,p)$-Poincaré-inequality immediately improves to a $(1, p - \epsilon)$-Poincaré-inequality [86]. One might ask if such self-improvement can be pushed to show that more general Poincaré-type inequalities also self-improve.

By a Poincaré type-inequality we will refer to inequalities that control the oscillation of a function by some, possibly non-linear, functional of its gradient. Such inequalities have appeared in the work of Semmes [130], Bate and Li [12]

and Jana Björn in [15]. Results in this spirit have appeared in [47]. We will here show that on geodesic doubling metric measure spaces many types of Poincaré-inequalities imply a true Poincaré-inequality. In particular, most of the definitions of Poincaré-inequalities produce the same category of PI-spaces.

First, we introduce what we mean by a non-homogeneous Poincaré-inequality.

Definition 1.2.43. Let (X, d, μ) be a metric measure space and $C \geq 1$ some constant. Let $\Phi, \Psi \colon [0, \infty) \to [0, \infty)$ be increasing functions with the following properties.

- $\lim_{t \to 0} \Phi(t) = \Phi(0) = 0$

- $\lim_{t \to 0} \Psi(t) = \Psi(0) = 0$

We say that (X, d, μ) satisfies a non-homogeneous (Φ, Ψ, C)-Poincaré-inequality if for every 2-Lipschitz[3] function $f \colon X \to \mathbb{R}$, every upper gradient $g \colon X \to \mathbb{R}$ and every ball $B(x, r) \subset X$ we have

$$\fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq r \Psi \left(\fint_{B(x,Cr)} \Phi \circ g \, d\mu \right).$$

This class of Poincaré-inequalities subsumes the ones considered by Björn in [15] and the Non-homogeneous Poincaré-Inequalities (NPI) considered by Bate and Li in [12]. We get the following theorem and corollary.

Theorem 1.2.44. *Suppose that (X, d, μ) is a quasiconvex D-doubling metric measure space and satisfies a non-homogeneous (Φ, Ψ, C)-Poincaré-inequality. Then*

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[3]The constant 2 is only used to simplify arguments below. Any fixed bound could be used. Also, by adjusting the right-hand side by scaling with the Lipschitz constant of f, the inequality could be made homogeneous.

(X, d, μ) is (C, δ, ϵ)-connected and moreover admits a true $(1, q)$-Poincareé inequality for some $q > 1$. All the variables are quantitative in the parameters. In particular, a non-homogeneous Poincaré-inequality in the sense of [12] implies a true Poincaré-inequality.

Remark: One could weaken the assumption of quasiconvexity if we allow any function f instead of Lipschitz functions, or by replacing the inequality with one over curve fragments. These variants are similar to the discussions in [12, 50]. As a corollary, we obtain the following result strengthening the conclusion of Dejarnette [47].

Corollary 1.2.45. *Suppose (X, d, μ) is a doubling metric measure space satisfying a weak $(1, \Phi)$-Orlicz-Poincaré-inequality in the sense of [15], then it satisfies also a $(1, q)$-Poincaré-inequality for some q.*

Recall, that Orlicz-Poincaré-inequalities are defined as follows.

Definition 1.2.46. Let (X, d, μ) be a metric measure space and $C \geq 1$ a constant. Assume Φ is a so-called Young-function, i.e. a convex increasing function $\Phi \colon \mathbb{R}_+ \to \mathbb{R}_+$, s.t.

$$\lim_{t \to 0} \frac{\Phi(t)}{t} = 0, \lim_{t \to \infty} \frac{\Phi(t)}{t} = \infty. \tag{1.2.47}$$

By convention set $\Phi(0) = 0$. The space X is said to satisfy a $(1, \Phi)$-Orlicz-Poincare inequality if for ever Lipschitz function f and upper gradient g for f, and every ball $B(x, r)$ we have

$$\fint_{B(x,r)} |f - f_B| \, d\mu \leq C r \Phi^{-1} \left(\fint_{B(x,r)} \Phi(g) \, d\mu \right).$$

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As a corollary of this we generalize Lemma 4.2 from [15] as follows. In particular we can derive new theorems even for classical spaces.

Corollary 1.2.48. *Suppose the space* $(\mathbb{R}^n, \mu)$ *satisfies some weak* $(1, \Phi)$-*Orlicz-Poincaré-inequality and that* μ *is D-doubling. Then* $(\mathbb{R}^n, \mu)$ *also satisfies a* $(1, q)$-*Poincaré-inequality for some* $q > 1$.

We remark, that this is a "self-improvement" result only in the sense that we understand Poincaré-inequalities to be stronger than Orlicz-Poincaré-inequlities. However, some Orlicz-Poincaré-inequalities lie in between Poincaré-inequalities in that they are both weaker than some Poincaré-inequalities and stronger than others. This can be seen from [143, Corollary 4.2] and [47, Lemma 3.18]. We merely state that an Orlicz-Poincaré-inequality always implies *some* Poincaré-inequality, which may possibly be quite weak. The price of this generality is that for particular examples of Φ our bounds for the corresponding exponent are worse than the finer results in [47].

Finally, we will show the following theorem concerning A_∞-weights on metric measure spaces. This generalizes, when combined with [78], some aspects of the work in [56] concerning sub-Riemannian metrics and vector fields satisfying the Hörmander-condition. For definitions of the Muckenhoupt class $A_\infty(\mu)$, see Definition 1.2.33.

Theorem 1.2.49. *Let* (X, d, μ) *be a geodesic D-doubling metric measure space. If* (X, d, μ) *satisfies a* $(1, p)$-*Poincaré-inequality and* $\nu = w\mu$ *for some* $\nu \in A_\infty(\mu)$, *then there is some* $q > 1$ *such that* (X, d, ν)-*satisfies a* $(1, q)$-*Poincaré-inequality.*

Sometimes Sobolev-inequalities are more desired, and such inequalities could be derived using classical techniques [68]. It was remarked to us by Pekka Koskela

that more classical means would also lead to proving Theorem 1.2.49. However, this has not been published and here this result follows easily from our main theorem. We also remark, that the bounds for the exponent q in the previous theorem can be made explicit but remain quite bad. Even the classical methods alluded to by Pekka Koskela do not seem to provide better bounds for q, and it may be that examples exist that preclude much better bounds.

1.2.4 Differentiability results

Recall the following classical theorem due to Rademacher concerning Lipschitz functions on Euclidean space [117].

Theorem 1.2.50. *(Rademacher's theorem [117]) Let $f \colon \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function. Then for almost every $x \in \mathbb{R}^n$ there exists a unique $\nabla f(x) \in \mathbb{R}^n$, such that*

$$f(y) - f(x) = \nabla f(x) \cdot (y - x) + o(|y - x|). \tag{1.2.51}$$

The notation $o(h)$ is used for an arbitrary function with

$$\lim_{h \to 0} \frac{o(h)}{|h|} = 0.$$

The definition is easy to generalize for manifolds, except the derivative must depend on the charts. That is, Equation 1.2.51 becomes

$$f(y) - f(x) = df_i(\phi_i(y) - \phi_i(x)) + o(|y - x|). \tag{1.2.52}$$

where df_i is the differential, which exists almost everywhere, in the co-ordinates

of a chart $\phi_i \colon U_i \to R^n$. This expression is purely metric. Thus it makes sense to consider it on a metric space, and assume f, ϕ_i are Lipschitz functions.

We remark, that it is far from clear that such a notion would be of any interest in a general metric context. Any given definition might restrict the structure of the space so much that its study is of no consequence. Thus, for the moment, the reader might consider this definition a curiosity. Its interest lies in how it connects different ideas and leads to relevant questions on the structure of the underlying space.

Definition 1.2.53. A metric measure space (X, d, μ) is called a differentiability space (of analytic dimension $\leq M$) if there exist U_i and Lipschitz functions $\phi_i \colon U_i \to \mathbb{R}^{n_i}$ (with $n_i \leq M$) such that $\mu(N) = 0$,

$$X = \bigcup_i U_i \cup N,$$

and such that for any Lipschitz function $f \colon X \to \mathbb{R}$, for every i and μ-almost every $x \in U_i$, there exists a *unique* linear map $D^{\phi_i} f(x) \colon \mathbb{R}^{n_i} \to \mathbb{R}$ such that

$$f(y) = f(x) + D^{\phi_i} f(x)(\phi_i(y) - \phi_i(x)) + o(d(x, y)). \qquad (1.2.54)$$

The pairs (U_i, ϕ_i) are referred to as charts and the functions $\phi_i \colon U_i \to \mathbb{R}^{n_i}$ are called chart functions. The totality $\mathcal{U} = \{(U_i, \phi_i)\}$ is referred to as a measurable differentiable structure.

This definition is essentially already contained in [28] but the terminology is from [85]. There are two remarks. First of all, the uniqueness does not play a significant role in the theorem. If the derivative were not unique, a smaller dimensional chart would work. However, the uniqueness makes analysis of the

space easier and thus is included.

Another remark is that the implicit limit $y \to x$ in Equation 1.2.54 is a true limit. That is, $y \to x$ in the entire X and not a full measure subset. This precludes taking a measure μ supported on a smooth surface $S \subset \mathbb{R}^3$. It also forces much stronger regularity properties for the measure. In a way, it controls "every direction" on the space and not just almost every direction.

Finally, note that U_i need only be measurable subsets, and ϕ_i merely Lipschitz functions. There is no need for U_i to be open or for ϕ_i to be homeomorphisms. In some cases, such as for manifolds, this is the case, but generally it can fail. See the next example for a canonical example of this behavior.

Example: Laakso Diamond Let $X_0 = [0, 1]$. Define X_1 to be a graph with six vertices $v_1, v_2, v_3^{\pm}, v_4, v_5$, and edges e_i connecting v_i to v_{i+1} for $i = 1, 4$ and four edges $e_2^{\pm}, e_3^{\pm}$ connecting v_2 to $v_3^{\pm}$ and $v_3^{\pm}$ to v_4 respectively. See Figure 1.1. Equipt X_2 with the path-metric by declaring each edge to be of length 4^{-1}, and define a projection $\pi_1 \colon X_1 \to X_0$ by sending $v_i, v_i^{\pm} \to (i-1)/3$, and extending linearly. This map is 1-Lipschitz. Define a measure μ_2 on X_2 by taking the Lebesgue measure on e_1, e_4 and half of the Lebesgue measure on $e_2^{\pm}, e_3^{\pm}$. Define inductively metric measure graphs X_{i+1} with measures μ_{i+1} by replacing each edge e of X_i by a copy of X_1 scaled by a factor of 4^{-i}, and re-scaling the measure in a way such that the total measure remains unity. One can define a natural 1-Lipschitz projection $\pi_{i+1} \colon X_{i+1} \to X_i$. A metric measure space $(X_\infty, d, \mu_\infty)$ can be defined as either the Gromov-Hausdorff limit, or the inverse limit (see [35]), of the sequence X_i. It will be a PI-space by results of [35] (or alternatively, by constructing pencils of curves similar to [92]).
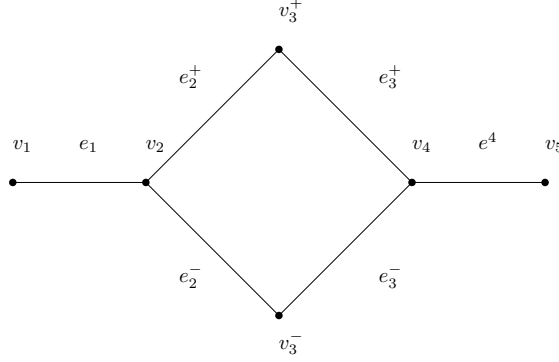
Figure 1.1: The first step of a Laakso diamond construction.

There is a natural map $\pi_\infty\colon X_\infty \to [0,1]$, and we can show that the single chart (X_∞, π_∞) is a measurable differentiable structure, and that X_∞ is therefore a differentiability space. This could be proven directly, or it can be seen as a consequence of Cheeger's Theorem 1.2.55 below. Note that π_∞ is not a homeomorphism. In fact, at generic points $x \in [0,1]$, $\pi_\infty^{-1}(x)$ is infinite. For more discussion on this construction see [94, p. 290] and [35, 36, 127]. The name of Laakso diamond is motivated by a similar constructions in [92] and [93]. See also section 1.1 for another construction.

Remark: The dimensions n_i between different charts in a differentiability space may vary. For example, one can take $X = \mathbb{R}^2 \cup_0 \mathbb{R}$, where the plane and line are glued along the origin. The space X will be covered by two charts $(\mathbb{R}^2, (x, y))$ and $(\mathbb{R}, t)$, where the chart maps are given by the natural co-ordinates. More complex examples can be constructed. In fact, by the thickening construction used to prove Theorem 1.2.63 and its Corollary 1.2.64, if $K \subset X$ is a compact subset of a PI-space X, then one can glue a tree T to it to make it a PI-space (while keeping the diameter comparable). The set K may have arbitrary analytic dimension in the aforementioned sense, but the tree is analytically 1-dimensional. The glued space

will not only be a differentiability space, but also a PI-space. Thus, the dimension can vary in rather complicated ways.

Theorem 1.2.55. *(Cheeger, [28]) If (X, d, μ) is a PI-space, then it is a differentiability space. The analytic dimension can be controlled in terms of the doubling constant and exponent in the Poincaré-inequality.*

Since this theorem was initially the only source of differentiability spaces, many authors began to suspect that differentiability spaces in some sense were equivalent to PI-spaces. However, observe the following example. Recall that if (X, d, μ) is a metric measure space and $A \subset X$ a Borel subset, we can define a restricted distance function $d_A = d|_{A \times A}$ and a restricted measure given by $\mu_A(B) = \mu(A \cap B)$.

Example: Positive measure subsets Let (X, d, μ) be a differentiability space and $A \subset X$ a Borel set with $\mu(A) > 0$, then (A, d_A, μ_A) is also a differentiability space, where d_A, μ_A are the restricted measure and distance. This may seem trivial, but entails a subtle point. For simplicity assume that there is a chart (U_i, ϕ_i) such that $A \subset U_i$. Then, for every Lipschitz function $f \colon A \to \mathbb{R}$, we can find a Lipschitz function $\overline{f} \colon X \to \mathbb{R}$ by MacShane Lemma 1.2.7 with $\overline{f}|_A = f$. By Equation 1.2.54, for almost every $x \in A$ we have

$$f(y) = f(x) + D^{\phi_i} f(x)(\phi_i(y) - \phi_i(x)) + o(d(x, y)), \qquad (1.2.56)$$

where $y \in X$. Take $y \in A$, and we get the same limit on A. However, the derivative $D^{\phi_i} f(x)$ such that the limit holds on A may no longer be unique. To ascertain uniqueness we would need to assume that A is not porous in X. Porosity is defined in Definition 4.1.5. If X is doubling, this would follow from being of positive

measure. However, in general, X may fail to be doubling. By a result of Bate and Speight [14], X is asymptotically doubling and the same result can be shown. See also [12] for a similar discussion.

The previous discussion shows, essentially, that the property of being a differentiability space is invariant under taking subsets. This is not true for Poincaré-inequalities. In fact, any PI-space is C-quasiconvex for some C. However, a positive measure subset may be totally disconnected and thus fail to possess a Poincaré-inequality. In general, the property of possessing some Poincaré-inequality is a rather subtle question even for subsets of Euclidean space. See for example [90].

Thus, the question of the relationship between differentiability spaces and PI-spaces is more delicate. For subsets of a PI-space two properties hold: their tangents possess Poincaré-inequalities, and they can be covered by subsets of PI-spaces (i.e. itself). We may ask if a Poincaré-inequality holds on a differentiability spaces in one of these senses. The first means that a Poincaré-inequality holds asymptotically. The second implies that a Poincaré-inequality holds up to decomposition and covering. This concept is made precise by the notion of PI-rectifiability.

Definition 1.2.57. A metric measure space (X, d, μ) is PI-rectifiable if there is a decomposition into sets U_i, N such that

$$X = \bigcup_i U_i \cup N$$

where $\mu(N) = 0$, and there exist distance preserving and measure preserving embeddings $\iota_i \colon U_i \to \overline{U}_i$ with $(\overline{U}_i, \overline{d}_i, \overline{\mu}_i)$ PI-spaces (with possibly very different constants).

The question of PI-rectifiability was posed explicitly by many authors, such as in [38]. Our main result shows that this indeed holds for a subclass of differentiability spaces. However, roughly concurrently to our work, Andrea Schioppa has shown that counter-examples exist to the general conjecture. We present next our main result, and contrast this with the main result of Schioppa.

The subclass of spaces our results is concerned with arises from the work of Kleiner and Cheeger. They studied how differentiability relates to embeddability questions. First, Cheeger noted in [28], that if a PI-space admits a bi-Lipschitz embedding to $\mathbb{R}^n$, then it is in fact rectifiable. Here rectifiable means, that the space can be covered up to a negligible set by positive measure subsets U_i which are bi-Lipschitz to subsets of Euclidean spaces.

On the other hand, there are several examples of PI-spaces that are not rectifiable (see the examples of Laakso diamond and graph above). This argument shows that such spaces can not be bi-Lipschitz embedded to Euclidean space. Thus, differentiability can be used to determine whether or not a space admits certain bi-Lipschitz embeddings.

Motivated by certain questions by Assaf Naor and others on embeddability questions in infinite dimensions, Cheeger and Kleiner studied embeddability in L^1 and so called RNP-Banach spaces by using similar arguments from differentiation. See the sequence of papers [33, 34, 35, 36, 37]. The class of RNP-Banach spaces can be defined in many ways, but we use the following definition.

Definition 1.2.58. A Banach space V has the Radon-Nikodym property if every Lipschitz function $f : [0, 1] \to V$ is almost everywhere differentiable. In other words, for almost every $x \in [0, 1]$ there exists a linear map $df_x \colon \mathbb{R} \to V$ such that

$$f(y) = f(x) + df_x(y - x) + o(|h|).$$

For several other equivalent and sufficient conditions consult for example [116] or the references in [32].

Similar to the definition of Fréchet derivatives we define RNP-differentiability spaces.

Definition 1.2.59. A metric measure space (X, d, μ) is a RNP-differentiability (of analytic dimension $\leq N$) space if there exist sets U_i and Lipschitz functions $\phi_i \colon U_i \to \mathbb{R}^{n_i}$ (with $n_i \leq N$) such that $\mu(N) = 0$,

$$X = \bigcup_i U_i \cup N,$$

and such that for any Banach space V with the Radon-Nikodym property and any Lipschitz function $f \colon X \to V$, for every i and μ-almost every $x \in U_i$ there exists a *unique* linear map $D^{\phi_i} f(x) \colon \mathbb{R}^{n_i} \to V$ such that

$$f(y) = f(x) + D^{\phi_i} f(x)(\phi_i(y) - \phi_i(x)) + o(d(x, y)).$$

Again, the pairs (U_i, ϕ_i) are referred to as charts, and their totality is called a RNP-differentiable structure.

Note that the Definition 1.2.58 gives immediately, that an interval $[0, 1]$ is a RNP-differentiability space with a global chart. The main theorem of Cheeger and Kleiner shows that this holds for any PI-space.

Theorem 1.2.60. *(Cheeger-Kleiner, [33]) If (X, d, μ) is a PI-space, then it is a RNP-differentiability space.*

A partial converse to this theorem, and thus a partial answer to the question of the relationship between PI-spaces and differentiability, was discovered by Bate and Li in [12]. They proved that every RNP-differentiability space is asymptotically doubling, and satisfied an asymptotic non-homogenous Poincaré-inequality [12, Theorem 4.3]. Further, by taking tangets they showed that such an inequality holds on almost every tangent [12, Proposition 6.3].

The main tool in the proofs of Bate and Li was an asymptotic version of our connectivity condition 1.2.32 (cf. [12, Theorem 3.8]. Our starting point was to study this connectivity condition in the abstract. We observed, that not only does their connectivity condition give a non-homogenous Poincaré-inequality (cf. [12, Lemma 4.2]), but it also gave a true Poincaré-inequality. This is the content of Theorem 1.2.38. Combining the results of 1.2.38 and [12, Lemma 4.2, Proposition 6.3] we obtained the following improvement for Tangents of RNP-differentiability spaces. The statement is slightly technical due to the asymptotic doubling condition.

Corollary 1.2.61. *Let (X, d, μ) be an RNP-differentiability space. Then X can be covered by positive measure subsets V_i, such that each V_i is metric doubling, when equipped with its restricted distance, and for μ-a.e. $x \in V_i$ each space $M \in T_x(V_i)$ admits a PI-inequality of type $(1, p)$ for all p sufficiently large.* [4]

We prove this statement directly in section 4.3. It also follows from our main theorem below.

Bate and Li provide a method to decompose a RNP-differentiability space (X, d, μ) into parts with a a uniform and asymptotic form of Definition 1.2.32. The previous result on tangents is based on taking a tangent at such a point. We

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[4]Recall the discussion after Definition 1.2.11. The subset V is equipped with the restricted measure and metric.

give a new construction that "thickens" these sets to be true PI-spaces and thus resolves the question of PI-rectifiability for RNP-differentiability spaces.

Theorem 1.2.62. *A complete metric measure space (X, d, μ) is a RNP-Lipschitz differentiability space if and only if it is PI-rectifiable and every porous set has measure zero.*[5]

In other words the study of RNP-differentiability spaces reduces to that of PI-spaces.

The proof of Theorem 1.2.62 involves two steps. First the decomposition of Bate and Li is used to identify good subsets with appropriate doubling and connectivity behavior (Definitions 1.2.13 and 1.2.32). We need to interpret these conditions in a relative way. One the one hand a new, larger, candidate space which is well-connected should be constructed and that satisfies the connectivity condition 1.2.32 intrinsically. On the other hand, the resulting space needs to be shown to satisfy a Poincaré-inequality. The second objective is reached by using Theorem 1.2.38 that reduces a Poincaré-inequality to the (C, δ, ϵ)-connectivity property which is easier to demonstrate.

The following theorem resolves the problem of constructing the space. The relevant conditions are defined in chapter 4.

Theorem 1.2.63. *Let $r_0 > 0$ be arbitrary. Assume (X, d, μ) is a metric measure space and $K \subset A \subset X$ and K compact. Assume that X is (D, r_0)-doubling along A, A is uniformly $(\frac{1}{2}, r_0)$-dense in X along K, and A with the restricted measure and distance is $(C, 2^{-60}, \epsilon, r_0)$-connected[6] along K. There exists constants $\overline{C}, \overline{\epsilon}, \overline{D} > 0$,*

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[5]The assumption of porous sets is somewhat technical and is similar to the discussion in [13].

[6]By an iteration argument similar to Theorem 1.2.40, or as presented in [12], the constant 2^{-60}-could be replaced by any $\delta < 0$, but this would make our proof more technical and is unnecessary.

and a complete metric space $\overline{K}$ which is $\overline{D}$-doubling and locally $(\overline{C}, \frac{1}{2}, \overline{\epsilon}, r_0 2^{-20})$-connected, and an isometry $\iota \colon K \to \overline{K}$ which preserves the measure. In particular, the resulting metric measure space $\overline{K}$ is a PI-space.

This result can be referred to as a "thickening" construction because it entails enlarging a space into a PI-space. At the core of the proof of this statement is a way of discretizing a neighborhood of A in K, associating a type of graph to it and gluing the said graph to K. The graph is constructed similar to a hyperbolic filling [21, Section 2] (see also a more recent presentation in English [19]). It can be attached to K since K naturally sits at its boundary.

This tree-like structure is motivated by the goal of preserving doubling, and avoiding adding too many intervals at any location or scale. By gluing intervals on each scale, we provide a graphical approximation of the neighborhood of K in A. This ensures that any curve and curve-fragent in a neighborhood of K in A can be replaced by a curve fragment in the glued space. The replacement procedure is via a discretization of the path. Thus, the connectivity of the original space can be related to the connectivity of the glued space. The methods employed in this portion of the paper bear resemblance to the work of Schul and Azzam in [9].

To give a simpler idea of our construction, we remark, that the above theorem could be proven in an easier form if X is doubling. In such a case, we can set $A = X$, and get the following theorem.

Theorem 1.2.64. *Let $r_0 > 0$ be arbitrary. Assume (X, d, μ) is a metric measure space and $K \subset X$ and K compact. Assume X is (D, r_0)-doubling and $X is $(C, 2^{-60}, \epsilon, r_0)$-connected[7] along K. There exists constants $\overline{C}, \overline{\epsilon}, \overline{D} > 0$, and a*

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[7]By an iteration argument similar to Theorem 1.2.40, or as presented in [12], the constant 2^{-60}-could be replaced by any $0 < \delta < 1$, but this would make our proof more technical and is unnecessary.

complete metric space $\overline{K}$ which is $\overline{D}$-doubling and $(\overline{C}, \frac{1}{2}, \overline{\epsilon}, r_0 2^{-20})$-connected, and an isometry $\iota \colon K \to \overline{K}$ which preserves the measure. In particular, the resulting metric measure space $\overline{K}$ is a PI-space.

In contrast to our main result, Andrea Schioppa [6] was recently able to construct examples of differentiability spaces which can not be RNP-differentiability spaces. *A fortiori* these examples are not PI-rectifiable. In fact, his examples even allow differentiation of l^2-valued Lipschitz functions, and demonstrate that the structure of general differentiability spaces may significantly diverge from spaces with Poincaré-inequalities. His proof that such a space is a differentiability space involves new techniques from quantitative differentiation, and forces us to reconsider old problems on the local geometry of differentiability spaces. In particular, it means that our classification result can not be strengthened to cover larger classes of differentiability spaces.

On a more positive note, our classification result can be thought of as giving a description of PI-rectifiability spaces in terms of differentiability of infinite dimensional targets. The ability to classify the large class of such spaces by a fairly natural notion from first order analysis seems somewhat surprising.

1.3   Structure of Thesis

In the following three chapters we will present the proofs of the statements in the previous three sections. Chapter 2 is devoted to bi-Lipschitz embeddability questions. Chapter 3 covers Poincaré-inequalities and our quantitative connectivity condition. The final chapter 4 includes the application of the results from the third chapter to differentiability spaces. Each chapter contains some historical remarks

at the beginning, followed by terminology and proofs.

The theorems stated above will be re-stated in the proper context below. The restated theorems will be numbered according to where they first appeared in the thesis.

Chapter 2

Bi-Lipschitz embeddings for manifolds

2.1 Introduction

2.1.1 Previous work

It is an old problem to describe, in some insightful manner, the metric spaces which admit a bi-Lipschitz embedding into Euclidean space [70]. A necessary condition for embeddability is that (X, d) is metric doubling (see Definition 1.2.12). As a partial converse, Assouad proved that any doubling metric space admits an $\alpha-$bi-Hölder embedding into Euclidean space for any $0 < \alpha < 1$ [7] (also see [109], and the statement in 1.2.21). But due to counter-examples by Pansu [115] (as observed by Semmes [131]) and Laakso [92] additional assumptions are necessary for bi-Lipschitz embeddability.

Bi-Lipschitz embedding problems have been extensively studied for finite metric spaces, see [22, 76, 103]. The problem has usually been to study the asymptotics of

distortion for embedding certain finite metric spaces, either in general or restricting to certain classes of metrics (e.g., the earth mover distance). The image is either a finite dimensional l_p-space or an infinite dimensional Banach space such as L^1. Motivation for such embeddings stems, for example, from approximation algorithms relating to approximate nearest neighbor searches or querying distances. There has also been some study on bi-Lipschitz invariants such as Markov type, and Lipschitz extendability, which can be applied once a bi-Lipschitz map is constructed to a simple space [108].

For non-finite metric spaces the research is somewhat more limited. We mention only some of the contexts within which embedding problems have been studied: subsets of $\mathbb{R}^n$ with intrinsic metrics by Tatiana Toro [141, 142], ultrametric spaces [98, 99], certain weighted Euclidean spaces by David and Semmes [129, 131], and an embedding result for certain geodesic doubling spaces with bicombings [94] by Lang and Plaut. Some of these works concern bi-Lipschitz parametrizations, which is a harder problem. Concurrently with the a draft of this chapter Matthew Romney [122] has considered related embedding problems on Grushin type spaces and applied methods from [133] as well as the current thesis. The different abstract question of existence of any bi-Lipschitz parametrization for non-smooth manifolds has been discussed in [73]. In contrast to these previous results, our goal is to prove embeddability theorems for natural classes of smooth manifolds, with uniformly bounded distortion and target dimension.

The closest results to this work are from Naor and Khot, who construct embeddings for flat tori and estimate the worst possible distortion [87]. Improvements to their results were obtained in [69]. In comparison, our work provides embeddings for any compact or non-compact complete flat orbifold. Our bounds are arguably

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weaker due to the higher generality. Also, noteworthy is the paper of Bonk and Lang [18], where they prove a bi-Lipschitz result for Alexandrov-surfaces with bounded integral curvature. Related questions on bi-Lipschitz embeddability have been discussed in [5], where one considers an Alexandrov space target.

A number of spaces can be shown not to admit any bi-Lipschitz embedding into Euclidean space. In addition to classical examples such as expanders, nontrivial examples were found by Pansu [115] (as observed by Semmes [131]) and Laakso [92]. These examples are related to a large class of examples that are covered by Lipschitz differentiation theory, which was initially developed by Cheeger [28]. Related to this theorem, Cheeger and Kleiner have applied ideas from differentiation to study embeddability of spaces in L^1 and so called RNP-Banach spaces (see definition 1.2.58). See for example the series of papers [33, 34, 35, 36, 37].

2.1.2   Statement of results

We will re-state the results from sub-section 1.2.2. Most of the spaces considered in this chapter admit some bi-Lipschitz embedding but our interest is in proving uniform, or quantitative, embedding theorems where the distortion is bounded in terms of the parameters defining the space. While a number of authors have addressed the question of quantitative bi-Lipschitz embeddability, natural classes of spaces remain for which such embeddings haven't been constructed.

Inspired by a question originally posed by Pete Storm we focus on embedding certain classes of manifolds and orbifolds. In our discussion, we will always view Riemannian manifolds (and orbifolds) as metric spaces by equipping them with the Riemannian distance function. We emphasize that the problem of finding *quantitative* bi-Lipschitz embeddings of Riemannian manifolds into Euclidean spaces has

little to do with classical results such as the Nash isometric embedding theorem for Riemannian manifolds (cf. the examples in sub-section 1.2.2). The notion of isometric embedding in that context only requires the map to preserve distances infinitesimally, whereas a bi-Lipschitz map controls the distances at any scale up to a factor. A Nash embedding for a *compact* manifold/subset will, however, result in a bi-Lipschitz embedding with a possibly very large distortion. To give a bound for the distortion of our embedding we will need to resort to very different techniques. We also note, that Nash embeddings don't directly give bi-Lipschitz embeddings for non-compact spaces, while some of our results apply for such examples as well.

Manifolds comprise a large class of metric spaces and we need to place some assumptions in order to ensure uniform embeddability. In order to ensure doubling it is natural to assume a diameter bound as well as a lower Ricci-curvature bound. Our results require the somewhat stronger assumption of bounding the sectional curvature in absolute value. Thus, we are led to the following theorem.

Theorem 1.2.22. *Let $D > 0$. Every bounded subset A with $\mathrm{Diam}\,(A) \leq D$ in a n-dimensional complete Riemannian manifold M^n with sectional curvature $|K| \leq 1$ admits a bi-Lipschitz embedding $f \colon A \to \mathbb{R}^N$ with distortion less than $C(D, n)$ and dimension of the image $N \leq N(D, n)$.*

We emphasize that there are no assumptions on the injectivity radius or lower volume bound. If such an assumption were placed, the result would follow directly from Cheeger-Gromov compactness or a straightforward doubling argument (see Lemma 2.3.5 and [64]). While somewhat unexpected, we are also able to prove a version of the previous theorem for orbifolds.

Theorem 1.2.23. *Let $D > 0$. Every bounded subset A with $\mathrm{Diam}\,(A) \leq D$ in a*

n-dimensional complete Riemannian orbifold O^n with sectional curvature $|K| \leq 1$ admits a bi-Lipschitz embedding $f \colon A \to \mathbb{R}^N$ with distortion less than $C(D, n)$ and dimension of the image $N \leq N(D, n)$.

In particular, we have the following non-trivial result. It also allows for making the previous statement more concrete. We first need to define connected complete flat and elliptic orbifolds.

Definition 2.1.1. Let Γ be a discrete subgrop of the isometry group $\mathrm{Isom}(\mathbb{R}^n)$ $(\mathrm{Isom}(S^n))$. The quotient metric space $X = \mathbb{R}^n/\Gamma$ $(X = S^n/\Gamma)$ is called a connected complete flat (elliptic) orbifold.

There are other definitions, but for concreteness we use the previous definition. See [120, 139]. Note, that while the following theorem is a corollary, we will provide a direct proof for it, because it is needed to prove Theorem 1.2.22.

Theorem 1.2.24. *Every connected complete flat and elliptic orbifold O of dimension n admits a bi-Lipschitz mapping $f \colon O \to \mathbb{R}^N$ with distortion less than $D(n)$ and dimension of the image $N \leq N(n)$. The constants $D(n)$ and $N(n)$ depend only on the dimension n.*

Remark: We can give explicit bounds for the case of flat and elliptic orbifolds, but not for arbitrary bounded curvature orbifolds. This is because the proof of Fukaya's fibration theorem in [31] uses compactness in a few steps and thus does not give an explicit bound for the parameters $\epsilon(n), \rho(n, \eta, \epsilon)$. Some related work in [63], when combined with techniques from [31], is likely to give an explicit bound, but to our knowledge this has not been published. In any case, the bounds extracted by known means would grow extremely rapidly (see the bounds in [26, 124]). For

flat orbifolds and manifolds we attain a bound on the distortion $D(n)$ which is of the order $O(e^{Cn^4 \ln(n)})$.

For mostly technical reasons we prove an embedding result for certain classes of "model" spaces which occur in our proofs. These involve the notion of a quasiflat space which is a slight generalization of a complete flat manifold to the context of quotients of nilpotent Lie groups. They correspond to model spaces for collapsing phenomena. The precise definition is in Definition 2.3.17.

Theorem 2.1.2. *For sufficiently small $\epsilon(n)$, every $\epsilon(n)$-quasiflat n-dimensional orbifold M admits a bi-Lipschitz embedding into $\mathbb{R}^N$ with distortion and dimension N depending only on n. Further, every locally flat Riemannian orbivector bundle over such a base with its natural metric admits such an embedding.*

It may be of interest, that some of these spaces are not compact and do not have non-negative sectional curvature. Yet, they admit global bi-Lipschitz embeddings. This theorem could also be stated for orbifolds, and the proof below will directly apply to them. For terminology related to orbifolds we recommend consulting for example [48, 89, 120, 139]. Some of this terminology is covered in section **??**.

We remark that the results above can trivially be generalized to Gromov-Hausdorff limits of bounded curvature orbifolds, which themselves may not be orbifolds. The Gromov-Hausdorff limits of bounded curvature Riemannian manifolds have been intrinsically described using weak Alexandrov-type curvature bounds by Nikolaev in [112], and as such our results also lead to embeddings of such spaces.

2.1.3 Outline of method

Bi-Lipschitz embedding problems can be divided to two subproblems: embedding locally at a fixed scale and embedding globally. Similar schemes for constructing embeddings have been employed elsewhere, such as in [94, 110, 133]. Constructing global embeddings may be difficult, but by assuming a diameter bound and lower curvature bound we can reduce it to embedding a certain fine ϵ-net.

We are thus reduced to a local embedding problem at a definite scale. In particular, since we don't assume any lower volume bound, we need to apply collapsing theory in order to establish a description of manifolds at small but fixed scales. For general manifolds we first use a result from Fukaya [61] (see also [31]). Recall, that if g is a metric tensor of a Riemannian manifold, and T is any tensor, then its norm with respect to the metric is denoted by $||T||_g$. There are different tensor norms, but all of them are quantitatively equivalent. To fix a convention, $||T||_g$ will refer to the operator norm.

Theorem 2.1.3. *(Fukaya, [31, 60, 123]) Let (M, g) be a complete Riemannian manifold of dimension n and sectional curvature $|K| \leq 1$, and $\epsilon > 0$ be an arbitrary constant. For every $\eta > 0$, there exists a universal $\rho(n, \eta, \epsilon) > 0$ such that for any point $p \in M$ there exists a metric g' on the ball $B_p(10\rho(n, \eta, \epsilon))$ and a complete manifold M' with the following properties.*

-

$$||g - g'||_g < \eta$$

- *$(B_p(5\rho(n, \eta, \epsilon)), g')$ is isometric[1] to a subset of a complete Riemannian man-*

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[1]By isometry we mean Riemannian isometry in the sense that the metric tensors are transformed to each other via the push-forward of the differential. What we really use is that the ball $(B_p(\rho(n, \eta, \epsilon)), g')$ will have a distance preserving mapping to a subset of M'.

ifold M′.

- *M′ is either a ε-quasi flat manifold or a locally flat vector bundle over a ε-quasiflat manifold.*

The constant ϵ will always be very small. We will set $\epsilon = \epsilon(n)$ for some dimension dependent $\epsilon(n)$, which is fixed below in section 2.3. This choice guarantees that the model spaces are sufficiently nice. In other words their collapsing can be studied algebraically. Throughout the paper the constant $\epsilon(n)$ will have the same value, which depends on n. The constant η will be fixed as $1/4$, and it is used to guarantee that $(B_p(\rho(n, \eta, \epsilon)), g')$ is bi-Lipschitz to $(B_p(\rho(n, \eta, \epsilon)), g)$. Since ϵ and η are chosen fixed, dependent on dimension, the scale $\delta = \rho(n, \eta, \epsilon)$ at which we have a nice model space, depends only on the dimension.

This theorem isn't stated as such in the references. Our terminology is different and is introduced in Definitions 2.3.17 and 2.3.26. In [60] a topological version is stated and in [31] the main results concern a fibration structure.

The statement of this theorem is contained in a similar form in the appendix of [31, Appendix 1], and the proof of our version can be derived from it. The main difference is that in [31] the role of M' is played by the normal bundle of a singular orbit (of the N-structure), $\nu(\mathcal{O}_q)$, with a natural metric. Here we have merely explicated that such a normal bundle can be described as a a vector bundle over a $\epsilon(n)$-quasiflat manifold. In section **??** we outline a proof of the analogous statement for orbifolds. We also use the results in [123] that guarantee us a dimension independent bound on the sectional curvatures of the model spaces.

Fukaya's fibration theorem permits us to reduce the embedding problem to embedding certain vector bundles ("model spaces"). These spaces have the "twisted

product" structure of a vector bundle over a quasiflat base. Further, these spaces also have the advantage of possessing an algebraic description. Thus, their metric geometry can be studied using the additional structure. We next give a rough description of our approach to embedding the vector bundles M that arise. We first use an approximation argument from Lemma 2.4.11 and Lemma 2.3.6 to reduce the problem to embedding locally flat Riemannian vector bundles over flat manifolds. Next, the flat manifold M is decomposed into pieces, and embeddings for each piece are patched together using Lipschitz extension theorems. These decompositions are similar in spirit but were developed independently of arguments by Seo and Romney in [122, 133].

To be more explicit, assume that the diameter of the base of the vector bundle is unity. Let us assume $L = V(S)$ is a locally flat Riemannian vector bundle over a flat base S. The space S is also the zero-section of the vector bundle L equipped with the restricted metric. Define for $i \geq 1$ the sets T_i to be the points at distance roughly 2^i from the zero section. Additionally, define T_0 to be the points $x \in L$ with $d(x, S) \leq 4$. By using the radial function $r(x) = d(x, S)$, we are able to subdivide the embedding problem for the entire space to that of embedding each T_i individually.

To embed each T_i, we re-scale the space by a definite amount to reveal a simpler vector bundle structure at a small, but definite scale. This is done using an algebraic collapsing theory argument. The difficult aspect of the proof here is showing that these local descriptions exist at a scale comparable to the diameter of T_i, which is necessary for applying a doubling Lemma 2.3.5 [2]. Further, the nature

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[2]If T_i is scaled down to unit size, the curvature bound is scaled up by 2^{2i}, and thus applying collapsing theory is difficult. If one only wants to prove Theorem 1.2.22, then this scaling up could be avoided. In fact, the algebraic arguments below can be avoided by such an approach, but this would lead to other technicalities, as well as weaker results.

in which the scaling produces a simpler space is somewhat delicate. For $i = 0$, the resulting spaces are of the same dimension, but have "fewer collapsing scales", i.e. their collapsing occurs more uniformly. For $i \geq 1$, the local descriptions split into a product of an interval and a lower dimensional space. In either case, by induction we can assume that the simpler spaces embed, and apply the doubling Lemma 2.3.5 to combine said embeddings to an embedding of T_i.

The algebraic collapsing theory argument involves quantitative versions of the local group arguments applied in collapsing theory [61]. Many of these arguments were inspired by proofs in [120, 140]. The core insight is that, up to a finite index, the fundamental groups are lattices. Lattices have "nice" bases, and the lengths of these basis elements give essentially canonically defined collapsing scales.

We also construct embeddings for bounded curvature orbifolds. For such spaces we need a version of Fukaya's theorem for orbifolds. The proof of this result is essentially contained in [31, Appendix 1], and [48], but we provide a rough outline in section **??**.

Theorem 2.1.4. *Let (O, g) be a complete Riemannian orbifold of dimension n and sectional curvature $|K| \leq 1$, and $\epsilon > 0$ be an arbitrary constant. For every $\eta > 0$ there exists a universal $\rho(n, \eta, \epsilon) > 0$ such that for any point $p \in O$ there exists a metric g' on the ball $B_p(10\rho(n, \eta, \epsilon))$ and a complete Riemannian orbifold O' with the following properties.*

-

$$||g - g'||_g < \eta$$

- $(B_p(5\rho(n, \eta, \epsilon)), g')$ *is isometric to a subset of O'.*

- O' is either a ϵ-quasiflat orbifold or a locally flat Riemannian orbivector bundle over a ϵ-quasiflat orbifold S.

The same comments on the choice of these parameters apply as before. When we apply this theorem, we fix $\epsilon = \epsilon(n)$ and $\eta = 1/4$, which allows us to use that the scale of the model spaces $\rho(n, 1/4, \epsilon(n))$ only depends on the dimension.

The methods to embed bounded curvature orbifolds are essentially the same as for the manifold case, except for modifying terminology and adding some additional cases.

2.1.4 Open problems

A natural question is to further study the dependence of our results on the curvature bounds assumed. At first one might ask whether the upper curvature bound is necessary. We conjecture that it may be dropped. In fact, in light of our results on orbifolds, it seems reasonable to presume that Alexandrov spaces admit such an embedding. For the definition of an Alexandrov space we refer to [23, 24]. The conjecture is also interesting within the context of Riemannian manifolds.

Conjecture 2.1.5. *Let $D > 0$. Every bounded subset A with* Diam $(A) \leq D$ *in a n-dimensional complete Alexandrov space X^n with curvature $K \geq -1$ admits a bi-Lipschitz embedding $f \colon A \to \mathbb{R}^N$ with distortion less than $C(D, n)$ and dimension of the image $N \leq N(D, n)$.*

If we assume in addition that the space X^n is volume non-collapsed, the theorem follows from an argument in [1]. One of the main obstacles in proving embedding theorems for these spaces is the lack of theorems in the Lipschitz category for Alexandrov spaces. Most notably, proving a version of Perelman's stability theorem

would help in constructing embeddings [82]. Perelman claimed a proof without publishing it and thus it is appropriately referred to as a conjecture. See Definition 1.2.9 for the definition of the Gromov-Hausdorff-distance d_{GH}.

Conjecture 2.1.6. *Let X^n be a fixed n-dimensional compact Alexandrov space with curvature $K \geq -1$. Then, there exists an $0 < \epsilon_0$ (depending possibly on X^n) such that for any other Alexandrov space Y^n with*

$$d_{GH}(X^n, Y^n) < \epsilon_0,$$

we have a bi-Lipschitz map $f \colon X^n \to Y^n$ with distortion at most L (which may depend on X^n).

Both of these conjectures seem hard. But mostly for lack of counter examples, we also suggest that the lower sectional curvature bound could be weakened to a lower Ricci-curvature bound. Weakening the curvature assumption to a simple Ricci-curvature bound results in great difficulty in controlling collapsing phenomena. However, even without collapsing the conjecture is interesting and remains open. Thus, we state the following conjecture.

Conjecture 2.1.7. *Let A be a subset with Diam $(A) \leq D$ in a n-dimensional complete Riemannian manifold (M^n, g) with curvature $\mathrm{Ric}(g) \geq -(n-1)g$. Assume $\mathrm{Vol}(B_1(p)) > v$ for some $p \in A$. Then, there exists a bi-Lipschitz embedding $f \colon A \to \mathbb{R}^N$ with distortion less than $C(D, n, v)$ and dimension of the image $N \leq N(D, n, v)$.*

Finally, we remark on the problem of optimal bounds for our embeddings in the Theorem for flat and elliptic orbifolds 1.2.24. As remarked before, we obtain

a bound for the worst-case distortion $D(n)$ of the order $O(e^{Cn^4 \ln(n)})$. The main source of distortion is the repeated and inefficient use of doubling arguments at various scales, which result in multiplicative increases in distortion. In a related paper, Regev and Haviv improve the super-exponential upper bound from [87] and obtain $O(n\sqrt{\log(n)})$ distortion for n-dimensional flat tori [69]. Thus, it seems reasonable to suspect that the true growth rate of $D(n)$ is polynomial. As pointed out to us by Assaf Naor, this problem is also related to [5] because certain finite approximations to Wasserstein spaces arise as quotients of permutation groups.

2.1.5 Outline

In the next section we give explicit embeddings for three types of bounded curvature spaces. These examples illustrate the methods used to prove the embedding results. We will not explain all the details, as some of them are presented well in other references and will become more apparent in the course of the proof of the main theorem. Following this we collect some general tools and lemmas that will be used frequently in the proofs of the main results. Some of these are very similar to [94, 110]. Finally, in the fourth section we give full proofs of the main embedding theorems. That section proceeds by increasing generalities. First flat manifolds are embedded, followed by flat orbifolds, and quasiflat orbifolds. Ultimately the results are applied to Riemannian manifolds and orbifolds. The final section collects a few of the most technical results on quotients of nilpotent Lie groups and collapsed orbifolds.

2.2 Embedding some key examples

We will exhibit nearly explicit embeddings for some concrete examples. The examples were chosen to reflect some of the general techniques used to prove the main theorems. We will skip over some details that will be contained in later proofs.

2.2.1 Lens spaces

Take two distinct co-prime numbers $p, q \in \mathbb{N}$. Consider the lens space $L(p, q)$, which is defined as the quotient of $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \colon |z_1|^2 + |z_2|^2 = 1\}$ by an action of $\mathbb{Z}_p$. This action is defined by $t(z_1, z_2) \to (e^{2\pi t/pi} z_1, e^{2\pi tq/pi} z_2)$ for $t \in \mathbb{Z}_p$. Denote the equivalence class of an element (z_1, z_2) by $[(z_1, z_2)]$. The action is free and isometric, wherefore $L(p, q)$ inherits a constant curvature metric. For an insightful discussion of these spaces see [140]. We ask for a bi-Lipschitz embedding $f \colon L(p, q) \to \mathbb{R}^N$ for some N, and distortion, independent of p and q.

Collapsing theory states that the space can be covered by charts in which it resembles a normal bundle over some simpler space. For the space $L(p, q)$, such charts can be realized explicitly by the sets $U_1 = \{[(z_1, z_2)] \in L(p, q) \colon |z_2| < \sqrt{3}|z_1|\}$ and $U_2 = \{[(z_1, z_2)] \in L(p, q) \colon |z_1| < \sqrt{3}|z_2|\}$. These two sets cover $L(p, q)$. Next, we will describe their geometry.

The sets U_j, for $j = 1, 2$, contain the "polar" circles S_j defined by $|z_j| = 1$. Further, U_j can be expressed as normal bundles over S_j with a curved metric. On the lift of U_1 to S^3 we can locally introduce co-ordinates (α, θ, ϕ) by $(\alpha, \theta, \phi) \to (\cos(\alpha)e^{i\theta}, \sin(\alpha)e^{i\phi})$ for $\alpha \in [0, \pi/3)$ and $(\theta, \phi) \in S^1 \times S^1$. The induced metric, becomes

$$g = |\sin(\alpha)|^2 |d\alpha|^2 + |\cos(\alpha)|^2 |d\theta|^2 + |\cos(\alpha)|^2 |d\alpha|^2 + |\sin(\alpha)|^2 |d\phi|^2$$
$$= |\cos(\alpha)|^2 |d\theta|^2 + |d\alpha|^2 + |\sin(\alpha)|^2 |d\phi|^2.$$

We can also define another metric on the lift of U_1 by

$$g_f = |d\theta|^2 + |d\alpha|^2 + \alpha^2 |d\phi|^2.$$

Note that the norms induced by g and g_f differ by a factor at most 4 (since $\sin(\alpha)^2 \geq \frac{1}{4}\alpha^2$ and $\cos(\alpha)^2 \geq \frac{1}{4}$ for $\alpha \in [0, \pi/3)$). The group $\mathbb{Z}_p$ acts on the lift of U_1 by isometries with respect to either of these metrics, and thus the latter descends to a metric g_f' on U_1, which is up to a factor 4 equal to the push-forward of the original metric g. In fact, (U_1, g_f') is isometric to a subset of the holonomy bundle, which is given by the $\mathbb{R}^2$-vector bundle over $\frac{2\pi}{p}S^1$ (the circle of radius $2\pi/p$), with holonomy generated by a $\frac{2\pi q}{p}$-rotation. Since the metric tensors differ only up to a factor and both (U_1, g) and (U_1, g_f') are geodesically convex, we can see that the identity map $\iota_1 \colon (U_1, g) \to (U_1, g_f')$ is bi-Lipschitz with factor 4. Similar analysis can be performed for U_2, where the bundle has holonomy generated by a rotation of angle $2\pi s/p$, where $sq \equiv 1 (\mathrm{mod}(p))$. One is thus led to consider the problem of embedding flat vector bundles with non-trivial holonomy. These spaces will be discussed in the next subsection, where we indicate a construction for the desired bi-Lipschitz maps $F_j \colon (U_j, g_f') \to \mathbb{R}^{n_j}$.

Finally, given the embeddings F_j, we define the map $f = (F_1, F_2, d(\cdot, U_1))$, where F_1 and F_2 are extended using the McShane lemma. To observe that this map is bi-Lipschitz, we refer to similar arguments in the proof of Lemma 2.3.5, or

the proof of [94, Theorem 3.2], which we present below. We remark, that there are many ways by which to patch up bi-Lipschitz embeddings from subsets into a bi-Lipschitz embedding of a whole. Our choice here is somewhat arbitrary. In fact, later in Lemma 2.3.5 a slightly different choice is employed.

For convenience of the reader, we restate an prove Theorem 3.2 from [94]. This result is not necessary for the proofs below as we will use slightly different arguments. However, understanding this argument is helpful.

Lemma 2.2.1. *[94, Theorem 3.2] Let X be a metric space, $A, B \subset X$ be two sets such that $X \subset A \cup B$ and $f \colon A \to \mathbb{R}^n$ and $g \colon B \to \mathbb{R}^m$ be two maps. If f and g are L-bi-Lipschitz and are extended arbitrarily to be L-Lipschitz on X, then $F(x) = (f(x), g(x), d(x, A))$ defines a CL-bi-Lipschitz map $F \colon X \to \mathbb{R}^{n+m+1}$ for some C, which depends only on L.*

Proof: Clearly F is $3L$-Lipschitz since its three components are L-Lipschitz. Thus, we only need to prove the lower bound in Equation 1.2.4. Let $x, y \in X$ be arbitrary. If we have $x, y \in A$ or $x, y \in B$, then

$$|F(x) - F(y)| \geq |f(x) - f(y)| \geq \frac{1}{L} d(x, y),$$

or

$$|F(x) - F(y)| \geq |g(x) - g(y)| \geq \frac{1}{L} d(x, y),$$

respectively. Thus, assume that both x and y don't lie in the same subset A or B. By symmetry, assume $x \in A$ and $y \notin A$. There are two cases, $10L^2 d(y, A) \leq d(x, y)$, or $d(x, y) < 10L^2 d(y, A)$. In the latter case

$$|F(x) - F(y)| \geq d(y, A) \geq \frac{1}{10L^2} d(x, y).$$

Consider next the first case. Define $y' \in A$ to be such that $d(y, y') \leq 2d(y, A)$. Then $|f(y) - f(y')| \leq Ld(y, y') \leq 2Ld(y, A) \leq \frac{1}{5L} d(x, y)$.

$$
\begin{aligned}
|F(x) - F(y)| &\geq |f(x) - f(y)| \\
&\geq |f(x) - f(y')| - |f(y') - f(y)| \\
&\geq \frac{1}{L} d(x, y) - \frac{1}{5L} d(x, y) \geq \frac{1}{2L} d(x, y).
\end{aligned}
$$

Combining the two lower bounds, and the upper bound, we get that F is $10L^2$-Bi-Lipschitz.

$\square$

2.2.2   Flat vector bundles

In the previous example, we reduced the problem of embedding a lens space to that of embedding flat vector bundles. We are thus motivated to consider the embedding problem for them. Consider the space E_θ^3, which is a flat $\mathbb{R}^2$-bundle over S^1 with holonomy θ. $E_\theta^3 = \mathbb{R} \times \mathbb{C}/\mathbb{Z}$, where the $\mathbb{Z}$-action is defined as $t(x, z) = (x + 2\pi t, e^{it\theta} z)$ for $t \in \mathbb{Z}$ and $z \in \mathbb{C}$. Let r be the distance function to the zero section, which is given by $r([x, z]) = |z|$. To embed this space, we use a decomposition argument. Let $T_0 = \{[x, z] \colon |z| \leq 4\}$ and $T_j = \{[x, z] \colon 2^{j-1} < |z| < 2^{j+1}\}$, for $j \geq 1$ an integer. Assume first that each T_j can be embedded with uniform bounds

on distortion and dimension by mappings $f_j\colon T_j \to R^n$. Then, we can collect the maps f_j with disjoint domains by defining four functions F_s for $s = 0, \ldots 3$ as follows. Consider the sets $D_s = \bigcup_{k=0}^{\infty} T_{s+4k}$, and define $F_s(x) = f_{s+4k}(x)$ for $x \in T_{s+4k}$. The resulting functions will be Lipschitz on their respective domains D_s. By an application of McShane extension Theorem 1.2.7 we can extend them to E_θ^3. The embedding we consider is $F = (r, F_0, F_1, F_2, F_3)$.

To see that $F(x)$ is bi-Lipschitz, we need to establish the lower bound in Equation 1.2.4. Take $x, y \in E_\theta^3$ arbitrary. First, if $d(x, y) \le 4|r(x) - r(y)|$, then the lower bound is trivial. Thus, assume $d(x, y) \ge 4|r(x) - r(y)|$. We will identify a j such that $x, y \in T_j$. This is obvious if $r(x) = r(y)$. We assume by symmetry that $r(x) > r(y)$ and $r(x) > 0$. Then, the lower bound is obtained by using the lower-Lipschitz bounds for f_j on T_j. We have the estimate $4|r(x) - r(y)| \le d(x, y) \le r(x) + r(y) + 1$. Thus, $r(y) \ge r(x) - |r(x) - r(y)| \ge 3/4 r(x) - 1/4 r(y) - 1/4$, and $r(y) \ge 3/5 r(x) - 1/5$. In particular, if $x \in T_0$, then $r(y) \le r(x) \le 4$, and thus $y \in T_0$. But, if $2^j \le r(x) \le 2^{j+1}$ for some $j \ge 2$, then $r(y) \ge 1/2 r(x) \ge 2^{j-1}$, and thus $x, y \in T_j$.

Next, we discuss the construction of the embeddings f_j. On T_0 the injectivity radius is bounded from below. Thus, we can find f_0 by patching up a finite number of bi-Lipschitz charts. More specifically, we use doubling and the Lemma 2.3.5. For T_j when $j \ge 1$, we will first modify the metric. Use co-ordinates $(x, r, \theta) \to (x, re^{i\theta}) \in \mathbb{R}^3$. The metric on T_j (or its lift in $\mathbb{R}^3$) can be expressed as

$$g = |dx|^2 + |dr|^2 + r^2 |d\theta|^2.$$

We change the metric to $g_f = |dx|^2 + |dr|^2 + 2^{2j}|d\theta|^2 = g_1$. As metric spaces, the space (T_j, g) is 10-bi-Lipschitz to (T_j, g_f). The new metric space (T_j, g_f) is

isometric to $(2^{j-1}, 2^{j+1}) \times T_\theta^j$, where T_θ^j is a torus defined by $\mathbb{R}^2/\mathbb{Z}^2$, where the $\mathbb{Z}^2$ action is given by $(n, m)(x, y) = (x + 2\pi n, y + n\theta + 2\pi m)$. The product manifold $(2^{j-1}, 2^{j+1}) \times T_\theta^j$ can be embedded by embedding each factor separately. Thus, the problem is reduced to embedding a flat torus. This can be done by choosing a short basis and is explained in detail in [87].

We comment briefly on the problem of embedding an arbitrary flat vector bundle. Consider a flat $\mathbb{R}^n$-bundle over S^1. The argument remains unchanged for T_0, but for T_j when $j \geq 1$ one can no longer modify the metric to be a flat product metric. Instead, one needs a further decomposition argument of the $\mathbb{R}^d$ factor similar to that used for lens spaces. If the base is more complicated than S^1, we are led to an induction argument where these decompositions are used to reduce the embedding problem for spaces that are simpler in some well-defined sense.

Finally, we highlight the main ideas used in the previous embedding constructions for lens spaces and holonomy-bundles. First, the space is decomposed into sets ("charts") on which we have in some sense a simpler geometry. The embeddings for these charts can be patched together to give an embedding of the space. In the case for lens spaces, the simpler geometry was that of a vector bundle. For vector bundles, the simpler geometry was that of a product manifold. The simpler geometries may need to be decomposed several times, but we can bound the number of iterated decompositions required by the dimension of the space. Ultimately, the problem is reduced to embedding something very simple such as a non-collapsed space with a lower bound on the injectivity radius. The main technical issues arise from the fact that high-dimensional spaces may require several decompositions and that for abstract spaces it is complicated to identify good

charts for which a simpler structure exists.

2.2.3 Quotients of the Heisenberg Group and Nilpotent groups

The Heisenberg group may be described as the simply connected Lie group of upper triangular 3×3-matrices with diagonal entries equal to one.

$$\mathbb{H} = \left\{ \left. \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \right| a, b, c \in \mathbb{R} \right\} \tag{2.2.2}$$

Define a lattice

$$\Gamma = \left\{ \left. \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \right| a, b, c \in \mathbb{Z}. \right\} \subset \mathbb{H}$$

which acts on $\mathbb{H}$ by left multiplication. We can define the quotient space $\mathbb{H}/\Gamma$. Next metrize this base as follows. Take a basis for the Lie algebra of left-invariant vector fields given by

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Consider the space $\mathbb{H}_\epsilon = (\mathbb{H}/\Gamma, g_\epsilon)$ with the metric defined by setting $X \perp Y \perp Z$, $|X| = |Y| = 1$ and $|Z| = \epsilon$. Initially, this metric is defined on $\mathbb{H}$, but since Γ acts by isometries the metric descends onto the quotient. A direct computation

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shows that the sectional curvatures of this space lie in

$$\left[-\frac{3\epsilon^2}{4}, \frac{\epsilon^2}{4}\right].$$

As $\epsilon \to 0$, the spaces $\mathbb{H}_\epsilon$ Gromov-Hausdorff converge to a torus T^2. The Gromov-Hausdorff approximation is given by the S^1-fibration map $\pi\colon \mathbb{H}_\epsilon \to T^2$ as follows,

$$\pi\colon \left[\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}\right] \to ([a],[b]) \in \frac{1}{2\pi}S^1 \times \frac{1}{2\pi}S^1 = T^2,$$

where $[a]$ is the fractional part of $a \in \mathbb{R}$. By $\frac{1}{2\pi}S^1$ we mean the circle of unit length. This map is easily seen to be well-defined on $\mathbb{H}_\epsilon$. We wish to find a bi-Lipschitz embedding for $\mathbb{H}_\epsilon$ with distortion independent of $\epsilon > 0$, as long as its sufficiently small.

An embedding for $\mathbb{H}_\epsilon$ can be visualized using Lemma 2.3.5. Consider a small ball $B_p(\delta) \subset T^2$, with some $\delta > 0$ independent of ϵ. Since T^2 is non-collapsed, there exists a $\delta > 0$ such that $B_p(\delta)$ is isometric to a ball in the plane. Then, since this ball is contractible, we can conclude that $\pi^{-1}(B_p(\delta))$ is diffeomorphic to $B_p(\delta) \times S^1$. Because the curvature is almost flat, we can show that this diffeomorphism can be chosen to be a bi-Lipschitz map with small distortion. This claim can be proven by a similar argument to Lemma 2.5.5. On the other hand, the space $B_p(\delta) \times S^1$ is easy to embed since it splits as a product of two simple spaces. The result can then be deduced by covering T^2, and thus $\mathbb{H}_\epsilon$, by a controlled number of these sets, and patching up the embeddings using Lemma 2.3.5 or a repeated application of Lemma 2.2.1.

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2.3 Frequently used results

Below, there are a number of constants determined from specific theorems. Since we wish to prove embedding theorems with quantitative bounds on the distortions, we need to be careful about the dependence on these parameters. In order to keep track of these different constants and functions throughout the chapter, we fix their meaning. The special constants are: $C'(n)$ from Lemma 2.3.16 and Definition 2.3.17, $c(n)$ is the maximum of the constants necessary for Lemmas 2.3.28, 2.3.16 and 2.3.14, $\epsilon(n)$ is fixed via Lemmas 2.3.23, 2.4.9 and 2.4.11, $\delta(n)$ from Lemma 2.5.5, and $C(n)$ from Lemma 2.5.4. Also, we will choose $c(n)$ increasing and $\epsilon(n)$ decreasing in n. Their values will be considered fixed throughout the chapter, although the precise value is left implicit. In general, we will use the convention that a constant $M(a_1, a_2, \ldots, a_n)$ is a quantity depending on $a_1, \ldots, a_n$ only.

2.3.1 Embedding lemmas

Consider any metric space X. As before, for any subset $A \subset X$ we call $N_\delta(A) = \{x \in A | d(x, A) < \delta\}$ its δ-tubular neighborhood. The metric spaces in this chapter will be complete manifolds M and orbifolds O equipped with a Riemannian distance function d.

We will use decomposition arguments in two ways. On the one hand we have spaces that admit certain splittings, such as cones and products.

Lemma 2.3.1. *Let (X, d_X), (Y, d_Y) be metric spaces and take $X \times Y$ with the product metric $d((x_1, y_1), (x_2, y_2)) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}$. If X and Y admit bi-Lipschitz embeddings $f \colon X \to \mathbb{R}^N$ and $g \colon Y \to \mathbb{R}^M$ with distortion L, then $X \times Y$ admits a bi-Lipschitz embedding $(f, g) \colon X \times Y \to \mathbb{R}^{N+M}$ with distortion at*

most $\sqrt{2}L$.

The proof of this result is trivial. The product space could also be equipped with different bi-Lipschitz equivalent metrics. In other cases the space admits a conical splitting. We define a cone over a metric space.

Definition 2.3.2. Let X be a compact metric space with $\text{Diam}(X) \leq \pi$. Then we define the cone over X as the space $C(X) = \{(t, x) \in \mathbb{R} \times X\} / \sim$, where $(0, x) \sim (0, y)$ for any $x, y \in X$, equipped with the metric

$$d([t, x], [s, y]) = \sqrt{t^2 + s^2 - 2ts \cos(d(x, y))}.$$

The metric cones which arise in our work will always be quotients of $\mathbb{R}^n$ by a group with a fixed point. Thus, we state the following lemma whose proof is immediate. Note that all the groups of the following lemma are forced to be finite.

Lemma 2.3.3. *Let Γ be a group acting properly discontinuously and by isometries on $\mathbb{R}^n$ which fixes $0 \in \mathbb{R}^n$. Then the group Γ acts properly discontinuously and by isometries on $S^{n-1} \subset \mathbb{R}^n$, and $\mathbb{R}^n/\Gamma$ is isometric to $C(S^{n-1}/\Gamma)$.*

Lemma 2.3.4. *Let X be a compact metric space with $\text{Diam}(X) \leq \pi$, and $Y = C(X)$. Further, assume that X admits a bi-lipschitz embedding to $\mathbb{R}^n$ with distortion L, then Y admits a bi-Lipschitz embedding to $\mathbb{R}^{n+1}$ with distortion $20L$.*

Proof: Let r be the radial function on Y, where points on Y are given by equivalence classes $[(r, x)]$ for $r \geq 0, x \in X$. Assume that $f : X \to \mathbb{R}^n$ is bi-Lipschitz with distortion L and $0 \in \text{Im}(f)$. Define $g(r, x) = (Lr, rf(x))$ on Y and show that it is the desired embedding. If (s, x) and (t, y) are points in Y and $s \leq t$, then

$$|g(s,x) - g(t,y)| \leq L|s - t| + s|f(x) - f(y)| + |s - t||f(y)|.$$

Note that $|f(y)| \leq L\pi$ (because Diam $(X) \leq \pi$). Further,

$$d((s,x),(t,y)) = \sqrt{t^2 + s^2 - 2ts\cos(d(x,y))} \geq 2sd(x,y)/\pi.$$

Using $s|f(x) - f(y)| \leq Lsd(x,y)$ we get

$$|g(s,x) - g(t,y)| \leq (L\pi + L + 2L/\pi)d((s,x),(t,y)) \leq 10Ld((s,x),(t,y)).$$

Next, we derive the necessary lower bound from Equation 1.2.4. Take any two points (s,x) and (t,y) in Y with $s \leq t$. Either $|s - t| \geq \frac{1}{20L^2}d((s,x),(t,y))$ or not. In the first case

$$|g(s,x) - g(t,y)| \geq L|s - t| \geq \frac{1}{20L}d((s,x),(t,y)).$$

In the latter case, $|s - t| \leq \frac{1}{20L^2}d((s,x),(t,y))$, and simple estimates give

$$
\begin{aligned}
|g(s,x) - g(t,y)| &\geq |g(t,x) - g(t,y)| - |s - t||f(x)| \\
&\geq \frac{1}{2L}d((t,x),(t,y)) - \frac{\pi}{20L}d((s,x),(t,y)) \geq \frac{1}{4L}d((s,x),(t,y)).
\end{aligned}
$$

These estimates complete the proof.

$\square$

Remark: Similar constructions would also apply for a spherical suspension of a metric space (see [24]) but we do not need that result.

When a simple splitting structure doesn't exist at a given scale, it may exist at a smaller scale. To enable us to accommodate for this we need the following "doubling argument". We emphasize that a metric space must be doubling in order to admit a bi-Lipschitz embedding into $\mathbb{R}^N$.

The following Lemma could be proven by repeatedly applying Lemma 2.2.1, but we give a different and slightly more direct argument.

Lemma 2.3.5. *(Doubling Lemma) Assume $0 < r < R, 0 < N, l$ are given and that X is a doubling metric space with doubling constant D. Fix a point $p \in X$. If for every point $q \in B_p(R)$ there is a l-bi-Lipschitz embedding $f_q \colon B_q(r) \to \mathbb{R}^N$, then there is a L-bi-Lipschitz embedding $G \colon B_p(R) \to \mathbb{R}^M$ for some $M > 0$. Further, we can bound the distortion of the bi-Lipschitz embedding G by $L \leq L(N, l, D, R/r)$ and the target dimension by $M \leq M(N, l, D, R/r)$.*

Proof: Construct a $r/8$-net $Q = \{q_i\}$ in $B_p(R)$ and let $f_i \colon B_{q_i}(r) \to \mathbb{R}^N$ be the bi-Lipschitz mappings assumed. The size of the net is $|Q| = H \leq D^{\log_2(R/r)+4}$. Translate the mappings so that $f_i(q_i) = 0$, and scale so that the Lipschitz-constant is 1. This will ensure that for any $a, b \in B_{q_i}(r)$ we have $|f_i(a) - f_i(b)| \geq 1/l^2 d(a, b)$. Further, let $F \colon x \to (d(x, q_i))_i \in \mathbb{R}^H$ be a distance embedding from the net. Group the q_i into K groups J_k such that if $q_i, q_j \in J_k$ then $d(q_i, q_j) > 4r$. The number of points in the net Q within distance at most $4r$ from any q_i is at most D^4. Thus, a standard graph coloring argument such as in [70] will furnish the partition J_k with K sets with $K \leq D^4$.

For each J_k, construct a map $g_k \colon \bigcup_{q_i \in J_k} B_{q_i}(r) \to \mathbb{R}^N$ by setting it equal to f_i on $B_{q_i}(r)$. Because the distance between the balls is at least $2r$, the Lipschitz constant

will not increase. To see this take arbitrary $a \in B_{q_i}(r)$ and $b \in B_{q_j}(r)$, where $i \neq j$ and $q_i, q_j \in J_k$. Then $d(a,b) \geq 2r$ and $|f(a) - f(b)| \leq |f(a) - f(q_i)| + |f(b) - f(q_j)| \leq 2r \leq d(a,b)$. Extend g_k using Lemma 1.2.7 to give a $\sqrt{N}$-Lipschitz map from the entire ball $B_p(R)$, and denote it by the same name. For the embedding combine these all into one vector

$$G = (g_1, \ldots, g_K, F).$$

Clearly this is $2\sqrt{N}D^4 + 2D^{\log_2(R/r)+4}$-Lipschitz. Also, it is a map $G\colon B(p,R) \to \mathbb{R}^M$ with $M = KN + H$. For the lower Lipschitz bound in (1.2.4) take arbitrary $a, b \in B_p(R)$. First assume $d(a,b) < r/2$. Then, there is a q_i such that $d(a, q_i) < r/8$ and $d(b, q_i) < r$. Both belong to the same ball $B_{q_i}(r)$ and for the k such that $q_i \in J_k$ we have $|G(a) - G(b)| \geq |g_k(a) - g_k(b)| = |f_i(a) - f_i(b)| \geq 1/l^2 d(a,b)$. Next assume $d(a,b) \geq r/2$. Then, there is a $q_i \in Q$ such that $d(a, q_i) < r/8 \leq d(a,b)/4$ and thus $d(b, q_i) \geq d(a,b) - d(a,b)/4 \geq 3/4 d(a,b)$. Thus, $|d(a, q_i) - d(b, q_i)| > d(a,b)/2$ and we get $|G(a) - G(b)| \geq |F(a) - F(b)| \geq d(a,b)/2$.

$\square$

Recall Definition 1.2.9 for the following statement.

Lemma 2.3.6. *(Gromov-Hausdorff-lemma) Let X be a doubling metric space with doubling constant D and let $\epsilon, \epsilon', l', l, m, N > 0$ be constants. Assume that Y is a metric space admitting a bi-Lipschitz map $h\colon Y \to \mathbb{R}^m$ with distortion l and that $d_{GH}(X, Y) \leq \epsilon'$. If for every $p \in X$ there is a l'-bi-Lipschitz embedding $f_p\colon B_p(\epsilon) \to \mathbb{R}^N$, then there is a L-bi-Lipschitz map $G\colon X \to \mathbb{R}^K$, with distortion $L \leq L(D, l, l', m, \epsilon'/\epsilon, N)$ and target dimension $K \leq K(D, l, l', m, \epsilon'/\epsilon, N)$.*

Proof: Fix $M = \epsilon'/\epsilon$. Let $g\colon X \to Y$ be a Gromov-Hausdorff approxima-tion. By Lemma 2.3.5 we can first construct for every $p \in X$ an embedding $f'_p\colon B_p(100l\sqrt{m}\epsilon') \to \mathbb{R}^{N'}$. By a scaling and dilation, we can assume that f'_p are 1-Lipschitz and $f'_p(0) = 0$. The distortion and the dimension N' will depend on l, l', D, N, m and M. Grouping f_p similarly to the proof in Lemma 2.3.5, we can define corresponding maps $g_k\colon X \to \mathbb{R}^N$ for $k = 1 \ldots H'$ for some H' depending on the parameters of the problem. Then, we can construct a map $F\colon X \to \mathbb{R}^{K'}$ with $F = (g_1, \ldots, g_H)$ with $K' = H'N$. Moreover, this construction guarantees that for every $x, y \in X$ with $d(x,y) \le 10l\epsilon'\sqrt{m}$ we have a $p \in X$ and an index i such that $x, y \in B_p(100l\sqrt{m}\epsilon')$ for some p and $g_i|_{B_p(100l\sqrt{m}\epsilon')} = f_p$. This can be used to give the lower bi-Lipschitz bound for Equation 1.2.4 for any pair $x, y \in X$ such that $d(x,y) \le 10l\epsilon'\sqrt{m}$.

Next, take a δ-net N_δ for $\delta = 4\epsilon'$. The map $h \circ g|_{N_\delta}\colon N_\delta \to \mathbb{R}^m$ is a $2l$-bi-Lipschitz map because g is 2-bi-Lipschitz on a $4\epsilon'$-net. Extend this to a map $H\colon X \to \mathbb{R}^m$ which is $2l$-bi-Lipschitz on N_δ and $2l\sqrt{m}$-Lipschitz on X. The map H is used to give a lower Lipschitz bound for pairs $x, y \in X$ with $d(x,y) > 10l\sqrt{m}\epsilon'$. The proof can now be completed by similar estimates as in Lemma 2.3.5 and defining $G(x) = (H(x), F(x))$. This gives a map $G\colon X \to \mathbb{R}^{K'+m}$.

$\square$

The main source of Gromov-Hausdorff approximants will be via quotients so we state the following Lemma.

Lemma 2.3.7. *Assume $(X, d_X), (Y, d_Y)$ are metric spaces and $\delta > 0$. If $\pi\colon X \to Y$ is a quotient map and if we define $d_Y(a,b) = d_X(\pi^{-1}(a), \pi^{-1}(b))$ for all $a, b \in Y$*

and Diam $(\pi^{-1}(a)) \leq \delta$ *for all* $a \in X$, *then* $d_{GH}(X, Y) \leq 2\delta$.

2.3.2 Group actions and quotients

Assume that X is a proper metric space, i.e. that $B_p(r)$ is precompact for every $p \in X$ and $r > 0$. Further, assume that Γ is a discrete Lie group acting on X by isometries. For any group, its identity element is denoted by either e or 0, depending on if it is abelian. For any $p \in X$, its *isotropy* group is denoted by $\Gamma_p = \{\gamma \in \Gamma | \gamma p = p\}$. We say that the action is *properly discontinuous* if for every $p \in X$ there exists a $r > 0$, such that $\{\gamma \in \Gamma | \gamma(B_p(r)) \cap B_p(r) \neq \emptyset\} = \Gamma_p$ and Γ_p is a finite group. We will study quotient spaces, which will be denoted by X/Γ, where the action of Γ is properly discontinuous. These spaces are also in some cases referred to as orbit spaces.

For $a \in X$ we will denote its orbit or equivalence class in X/Γ by $[a]$. Since the action is by isometries, we can define a quotient metric by

$$d([a], [b]) = \inf_{\gamma, \gamma' \in \Gamma} d(\gamma a, \gamma' b).$$

This distance makes X/Γ a metric space. For more terminology see [120, Chapter 5]. The global group action may be complicated, but the local action can often be greatly simplified. The following lemma is used to make this precise. First recall, that for any set of elements $S \subset G$ the smallest subgroup containing them is denoted by $\langle S \rangle$. We say that S generates $\langle S \rangle$. For a group G, we say that $\gamma_1, \ldots, \gamma_n$ are generators of the group, if $G = \langle \gamma_1, \ldots, \gamma_n \rangle$.

Lemma 2.3.8. *Let X be a metric space and Γ a discrete Lie group of isometries acting properly discontinuously on X. Take a point $p \in X$. Let $\Gamma_p(r) = \langle g \in$*

$\Gamma | d(gp, p) \leq 8r \rangle$ *be the subgroup generated by elements such that* $d(p, gp) \leq 8r$. *Then* $B_{[p]}(r) \subset Y = X/\Gamma$ *is isometric to* $B_{[p]'}(r) \subset X/\Gamma_p(r)$ *with the quotient metric.*

Proof: We will denote the cosets, functions and points in $Y' = X/\Gamma_p(r)$ using primes and corresponding objects in Y without primes. For example, elements of $X/\Gamma_p(r)$ can be represented by a $\Gamma_p(r)$-cosets $[x]'$, for $x \in X$, and elements of $Y = X/\Gamma$ will be represented as $[x]$. The action is still properly discontinuous and the distance function on the orbit space is given by $d'([x]', [y]') = \inf_{\gamma, \gamma' \in \Gamma_p(r)} d(\gamma x, \gamma' y)$. Since $\Gamma_p(r) \subset \Gamma$, we have a continuous map $F \colon (Y', d') \to (Y, d)$ which sends a $\Gamma_p(r)$-coset to the Γ-coset that contains it, i.e. $[x]' \to [x]$. We will next show that F maps $B_{[p]'}(r) = \{y' \in Y' | d'([p]', y') < r\}$ isometrically onto $B_{[p]}(r)$, from which the conclusion follows.

Let $[a]', [b]' \in B_{[p]'}(r) \subset X/\Gamma_p(r)$ and as before $F([a]') = [a], F([b]') = [b]$. Clearly $d([a]', [b]') \geq d([a], [b])$, as the Γ-cosets are super-sets of $\Gamma_p(r)$-cosets. We can choose the representatives of the cosets a, b such that $a, b \in B_p(r) \subset X$. By triangle inequality $d([a], [b]) < 2r$, and for any small $\epsilon > 0$ there is a $\gamma_\epsilon \in \Gamma$ such that $d(a, \gamma_\epsilon b) < d([a], [b]) + \epsilon < 2r$. But then $\gamma_\epsilon b \in B_p(3r) \subset X$ and $d(\gamma_\epsilon b, b) \leq 4r$. Therefore, $d(\gamma_\epsilon p, p) \leq d(\gamma_\epsilon p, \gamma_\epsilon b) + d(\gamma_\epsilon b, b) + d(b, p) \leq r + r + 4r \leq 6r$. Thus, $\gamma_\epsilon \in \Gamma_p(r)$ and therefore $d([a]', [b]') \leq d(a, \gamma_\epsilon b) \leq d([a], [b]) + \epsilon$. The result follows since ϵ is arbitrary.

Finally, the map is onto, because if $d([p], [a]) < r$, then we can choose the representatives such that $d(p, a) < r$. In particular $[a]' \in B_{[p]'}(r)$, and $F([a]') = [a]$ by definition.

$\square$

Motivated by the previous Lemma, we define a notion of a local group.

Definition 2.3.9. Let Γ be a discrete Lie group acting properly discontinuously on X. We denote by $\Gamma_p(r) = \langle g \in \Gamma | d(gp, p) \leq 8r \rangle$, and call it the local group at scale r (and location p).

In collapsing theory it is often useful to find invariant points and submanifolds. However, often it is easier to first construct almost invariant points/submanifolds, and then to apply averaging to such a construction to give an invariant one. For this reason, we will recall Grove's and Karcher's center of mass technique. See [67, 83] for a more detailed discussion and proofs.

Lemma 2.3.10. *(Center of mass Lemma) Let M be a complete Riemannian manifold with a two-sided sectional curvature bound $|K| \leq \kappa$ and $p \in M$ a point. Further, let μ be a probability measure supported on a ball $B_p(r)$ of radius $r \leq \min\left\{\frac{\pi}{2\sqrt{\kappa}}, \operatorname{inj}(M, p)/2\right\}$. There exists a unique point $C_\mu \in B_p(r)$ which minimizes the functional*

$$F(q) = \int d(q, x)^2 d\mu_x.$$

The minimizer C_μ is called the center of mass. The center of mass is invariant under isometric transformations of the measure. If $f \colon M \to M$ is an isometry, and $f_*(\mu)$ is the push-forward measure of μ, then

$$f(C_\mu) = C_{f_*(\mu)}.$$

As a corollary, one obtains the existence of fixed points for groups with small orbits. If G is any group acting on a manifold M, a point $p \in M$ is called a fixed point of a group action if $gp = g$ for all $g \in G$. A more traditional form of this lemma appears in [120, Lemma 5.9].

Corollary 2.3.11. *(Fixed point Lemma) Let M satisfy the same assumptions as in Lemma 2.3.10. Let a compact Lie group G act by isometries on M. If*

$$\operatorname{diam}(G_p) < \min\left\{\frac{\pi}{2\sqrt{\kappa}}, \operatorname{inj}(M,p)/2\right\},$$

then there exists a fixed point q of the G-action such that $d(q,p) \leq \operatorname{diam}(G_p)$.

In fact, one can use the Haar measure μ on G, and consider the push-forward measure $f_*(\mu)$ by the map $f\colon G \to M$ which is given by $f(g) = gp$. The center of mass $C_{f_*(\mu)}$ will be invariant under the group action by G.

We will repeatedly use an argument that allows us to derive metric conclusions from a stratification of a finite index subgroup. The idea of the statement is that a semidirect product with the stratified normal subgroup will itself admit a stratification. Such stratifications arise from Lemma 2.3.23 below. Assume Γ is a group. Then we call $|\cdot|\colon \Gamma \to \mathbb{R}$ a subadditive norm if the following hold.

- For all $g \in \Gamma$, $|g| \geq 0$ and $|e| = 0$.

- $|ab| \leq |a| + |b|$.

- $|a| = |a^{-1}|$

For groups acting on spaces, one can choose a natural class of subadditive norms.

Definition 2.3.12. Let Γ act on a metric space X, and let $p \in X$ be fixed. We define the subadditive norm at p for $g \in \Gamma$ as

$$|g|_p = d(gp, p).$$

If $g \in \text{Isom}(X)$ is an isometry, we denote $|g|_p = d(gp, p)$. Further, if Γ acts on a Lie group N, then we will often choose $p = e$ and denote $|\gamma| = |\gamma|_e$.

Lemma 2.3.13. *(Local Group Argument) Fix an arbitrary scale parameter $l > 0$. Let Γ be a group with a subadditive norm $|\cdot|$, and assume it admits a short exact sequence*

$$0 \to \Lambda \to \Gamma \to H \to 0,$$

where H is a finite group of size $|H| < \infty$. Assume $\Lambda_0 < \Lambda$ is a normal subgroup in Γ generated by all $g \in \Lambda$ of length $|g| \leq l$, and further assume that every element $g \in \Lambda \setminus \Lambda_0$ has length $|g| \geq 10|H|l$. Then the subgroup $\Gamma_0 < \Gamma$ generated by all $g \in \Gamma$ with $|g| \leq l$ admits a short exact sequence

$$0 \to \Lambda_0 \to \Gamma_0 \to H_0 \to 0,$$

where $H_0 < H$. Further, any element $g \in \Gamma_0$ is can be expressed as

$$g = \prod_{i=1}^{|H|+1} g_i \lambda,$$

where $|g_i| \leq l$ and $\lambda \in \Lambda_0$.

Proof: Define the group by generation: $\Gamma_0 = \langle g \in \Gamma | |g| \leq l \rangle$. Define S' to be the collection of left-cosets of Λ in Γ that contain a representative $s \in \Gamma$ with $|s| \leq l$. Define S to consist of a representative s for each left coset in S' with $|s| \leq l$. Clearly $|S| = |S'| \leq |H|$. Also, the cosets in S' coincide with Λ_0-cosets, because if $s = \lambda s'$ for some $\lambda \in \Lambda$ and $|s|, |s'| \leq l$, then $|\lambda| \leq |s| + |s'| \leq 2l$, and thus $\lambda \in \Lambda_0$.

Construct W using all the "words" w of length at most $|H|+1$ formed by mul-

tiplying elements s and s^{-1} for $s \in S$, including the empty word which represents the identity element of Γ. None of these words represent elements of $\Lambda \setminus \Lambda_0$ since for any $w \in W$ we have $|w| \leq (|H| + 1)l$. Denote by $W' = \{[w]_{\Lambda_0}|w \in W\}$ the left-cosets of Λ_0 represented by elements in W. Also, let $\overline{\Gamma} = \bigcup_{A \in W'} A \subset \Gamma$. We will show that $\overline{\Gamma} = \Gamma_0$. In other words, we show that W exhausts the left-cosets of Λ_0 in Γ_0, and that these cosets have a product given by the product of their representatives. To do so, it is sufficient to show that any product of two elements in W is equivalent modulo Λ_0 to an element in W. This shows that $\overline{\Gamma}$ is closed under products. Since also $\overline{\Gamma} < \Gamma_0$ as it is generated by $S \cup \Lambda_0$, we have $\Gamma_0 = \overline{\Gamma}$.

Take any word w' with length longer than $|H| + 1$. Consider the subwords $w'_1, \ldots, w'_{|H|}$ formed by the first $1, \ldots, |H| + 1$ letters of w', respectively. Because there are only $|H|$ cosets, two of them will be in the same Λ-coset in Γ. Thus, assume w_i and w_j with $i < j$ belong to the same coset. Let $w' = w_j v_j$, where v_j is the product of the remaining letters. We have $w_i w_j^{-1} \in \Lambda$. But also $|w_i w_j^{-1}| \leq (2|H| + 1)l$ so $w_i w_j^{-1} \in \Lambda_0$. Thus, we have for the Λ_0-cosets $[w_i]_{\Lambda_0} = [w_j]_{\Lambda_0}$, and thus $[w']_{\Lambda_0} = [w_j]_{\Lambda_0}[v_j]_{\Lambda_0} = [w_i]_{\Lambda_0}[v_j]_{\Lambda_0} = [w_i v_j]_{\Lambda_0}$. The word $w_i v_j$ is shorter than w'. This process can be continued as long as w' is longer than $|H| + 1$. Thus, any word longer than $|H| + 1$ is equivalent modulo elements in Λ_0 to an element in W. In particular, the cosets formed by products of cosets corresponding to elements in W can be represented by elements contained in W. This allows one to define a group structure for the cosets represented by W, and therefore a subgroup $H_0 < H$. It is immediate, that H_0 contains all the cosets generated by elements in S.

By the previous argument, Γ_0 is a finite extension of Λ_0 of index at most $|H_0|$. Also, Λ_0 is normal in Γ_0 since it is normal in Γ. The final statement about

representing elements of g as products follows from the above, since g belongs to a coset represented by an element in W.

$\square$

2.3.3   Nilpotent Lie groups, lattices and collapsing theory

Nilpotent geometries occur as model spaces for collapsing phenomena. Thus, we start by an estimate relating to the curvature of a general simply connected nilpotent Lie group N equipped with a left-invariant metric g. For definitions of nilpotent Lie groups as well as the computations involved here, see [45]. Associate to the Lie group N its nilpotent Lie algebra $\mathfrak{n}$. We construct a triangular basis for $\mathfrak{n}$ as follows. Let $\mathfrak{n}_0 = \mathfrak{n}, \mathfrak{n}_1 = [\mathfrak{n}, \mathfrak{n}], \ldots, \mathfrak{n}_{k+1} = [\mathfrak{n}, \mathfrak{n}_k]$. For k large enough $\mathfrak{n}_k = \{0\}$. Let X_i be an orthonormal basis, such that there are integers k_j and $X_i \in \mathfrak{n}_j$ for $i \geq k_j$ and $X_i \perp \mathfrak{n}_j$ for $i < k_j$. Let c_{ij}^k be the Maurier-Cartan structural constants with respect to this basis, i.e.

$$[X_i, X_j] = \sum_k c_{ij}^k X_k.$$

By construction of the basis, $c_{ij}^k \neq 0$ only if $k > j$. Computing the sectional curvatures $K(X_i, K_j)$ using [105] we get for $j \geq i$:

$$K(X_i, X_j) = \frac{1}{4} \sum_{k<j} \left(c_{ik}^j\right)^2 - \frac{3}{4} \sum_{k>j} \left(c_{ij}^k\right)^2.$$

As such, bounding the curvature is, up to a factor, the same as bounding the structural constants. Most of the time we will scale spaces so that sectional cur-

vature satisfies $|K| \leq 2$. Therefore, we will assume throughout this chapter that, with respect to any orthonormal basis, $|c_{ij}^k| \leq C'(n)$, where $C'(n)$ depends only on the dimension n. For comparison and later use we will mention the following theorem.

Theorem 2.3.14. *(Bieberbach, [25,120]) There exists a dimension dependent constant $c(n)$ such that the following holds. If M is a complete (not necessarily compact) connected flat Riemannian orbifold of dimension n, there exists a discrete group of isometries Γ acting on $\mathbb{R}^n$ such that $M = \mathbb{R}^n/\Gamma$. Also, there exists an affine k-dimensional subspace $O \subset \mathbb{R}^n$ which is invariant under Γ, and on which the action of Γ is properly discontinuous and co-compact. Further, there exists a subgroup $\Lambda < \Gamma$ with index $[\Gamma : \Lambda] \leq c(n)$ which acts on O by translations. Moreover, M is isometric to a locally flat Riemannian $n-k$-dimensional vector bundle over O (see Definition 2.3.26 below).*

Collapsing theory involves working with certain classes of spaces, which we define here. All notions in this chapter are defined for orbifolds. The versions for manifolds are obtained by a slight variation of terminology.

Definition 2.3.15. A compact Riemannian orbifold S is called $\epsilon-$almost flat if $|K| \leq 1$ and $\operatorname{Diam}(S) \leq \epsilon$.

Throughout this chapter ϵ will be small enough, i.e we will only consider $\epsilon(n)'$-almost flat n-dimensional manifolds for small enough $\epsilon(n)'$. The choice is dictated by Theorem 2.3.16 and Lemma 2.3.23. For such manifolds a detailed structure theory was developed by Gromov and Ruh ([25,26,65,124]). Later, Ding observed [48] that with minor modifications the proof as presented in [26, 124] generalizes

to orbifolds[3]. We state this structure theory here.

For definitions of orbifolds and some of the other terminology, we refer to the final section in this chapter and [89, 120, 139]. For any Lie group G we denote by $\text{Aff}(G)$ the Lie group of affine transformations of G, which preserve the flat left-invariant connection ∇_{can}. The left-invariant connection ∇_{can} is defined as the flat connection making left-invariant fields parallel. By $\text{Aut}(G)$, we denote the Lie group of its automorphisms. Throughout N will denote a simply connected nilpotent Lie group.

Theorem 2.3.16. *([26, 48, 124] Gromov-Ruh Almost Flat Theorem[4]) For every n there are constants $\epsilon'(n), c(n)$ such that for any $\epsilon'(n)$-almost flat orbifold S with metric g there exists a simply connected nilpotent Lie group N, a left-invariant metric g_N on N, and a subgroup $\Gamma < \text{Aff}(N)$ with the following properties.*

- *Γ acts on (N, g_N) isometrically, co-compactly and properly discontinuously. In particular N/Γ is a Riemannian orbifold.*

- *We have the bound $|K| \leq 2$ for the sectional curvature of N, and $||[\cdot, \cdot]||_{g_N} \leq C'(n)$ for the norm of the bracket.[5]*

- *There is an orbifold diffeomorphism $f \colon S \to N/\Gamma$, such that $||f_* g - g_N||_g < \frac{1}{2}$. (The bound here could be replaced by any $\delta < 1$, but we fix a choice to reduce the number of parameters in the theorem.)*

- *There exists a normal subgroup $\Lambda < \Gamma$, such that $[\Gamma : \Lambda] \leq c(n)$ and Λ acts on N by left-translations. In other words, by slight abuse of notation, $\Lambda < N$.*

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[3]The metric closeness part is not explicit in [48], but a direct consequence of applying [124].

[4]We also recommend consulting the preprint version of the paper by Ding which contains a more complete proof [49].

[5]For the precise sectional curvature bound, it is necessary to also use a remark in [123].

We remark that we consistently use N/Γ to indicate a quotient under a group action even though here Γ is acting on N on the left and the translational subgroup Λ acts on N by *left* multiplication. Again $|| \cdot ||_g$ denotes the operator norm of the various tensors. In the work of [48] and [124] no explicit bounds are derived for the Lie bracket and curvature, but as stated in [31] these follow from the methods and the curvature bounds at the beginning of this section.

In the course of the proof of the main theorem, we will need certain special classes of orbifolds that we will call quasiflat. These will be certain quotients of a nilpotent Lie group N by a group $\Gamma < \mathrm{Aff}(N)$. Each element $\gamma \in \mathrm{Aff}(N)$ can be represented by $\gamma = (a, A)$, where $a \in N$ and $A \in \mathrm{Aut}(N)$. The action of (a, A) on N is given by

$$(a, A)m = aAm$$

where $m \in N$. The component A is called *the holonomy* of γ. If A is reduced to identity, the operator can be expressed as (a, I), and is called translational. Translational elements $\gamma \in \Gamma$ can be canonically identified by an element in N, and we will frequently abuse notation and say $\gamma \in N$. There is a holonomy homomorphism

$$h \colon \Gamma \to \mathrm{Aff}(N)$$

given by $h(\gamma) = A$.

The Lie group N will always be equipped with a left-invariant metric g. As in the previous section, if $p \in N$ is given and $g \in \mathrm{Isom}(N)$, then we denote

$$|g|_p = d(gp, p).$$

Definition 2.3.17. A Riemannian orbifold S is called a *(ϵ-)quasiflat* orbifold if S is isometric to a quotient N/Γ of a simply connected nilpotent Lie group (N, g) equipped with a left-invariant metric g by a discrete co-compact group of isometries Γ satisfying the following.

- For the left-invariant metric g the sectional curvature is bounded by $|K| \leq 2$ and $||[\cdot, \cdot]||_g \leq C'(n)$.

- The group Γ acts on the space N isometrically, properly discontinuously and co-compactly.

- The quotient space N/Γ is a Riemannian orbifold with $\mathrm{Diam}\,(S) = \mathrm{Diam}\,(N/\Gamma) < \epsilon$. The space N is referred to as the (orbifold) universal cover and Γ is called the orbifold fundamental group.

We also define a flat orbifold as follows. Another definition, used in Theorem 2.3.14, requires that the space be complete and that the sectional curvatures vanish. This is, however, equivalent by the same theorem to the following.

Definition 2.3.18. If Γ is a group acting co-compactly, properly discontinuously and isometrically on $\mathbb{R}^n$, then we call $S = \mathbb{R}^n/\Gamma$ a flat orbifold.

The manifold versions of the previous definitions only differ by assuming that the actions are free. Recall, that a group action of Γ on a manifold M is *free* if the isotropy group is trivial for any $p \in M$, i.e. $\Gamma_p = \{e\}$, where e is the identity element of Γ.

By Theorem 2.3.16 any $\epsilon(n)'$-almost flat manifold is bi-Lipschitz to an $\epsilon(n)''$-quasiflat manifold with a slightly different $\epsilon(n)''$. Also, by Theorem 2.3.14 every compact flat manifold is also a flat orbifold in the previous sense. We will choose

$\epsilon(n)$ so small, that any $\epsilon(n)$-quasiflat manifold is $\epsilon(n)'$-almost flat, and so that (2.3.24) holds. Further, we will assume, by possibly making $\epsilon(n)$ smaller, that $\epsilon(n) < \epsilon(m)$ for $n > m$.

We first prove a result concerning the algebraic structure of Γ. This can be thought of as a generalization of Bieberbach theorem (see [25]).

Lemma 2.3.19. *Let $c(n)$ be the constant in Gromov-Ruh Almost Flat theorem 2.3.16. Let $S = N/\Gamma$ be a $\epsilon(n)$-quasiflat n-dimensional orbifold, then Γ has a finite index subgroup Λ with $[\Gamma : \Lambda] \leq c(n)$, and $\Lambda = N \cap \Gamma$, i.e. Λ acts on N by translations.*

Proof: Consider the left invariant metric g of N. Since a $\epsilon(n)$-quasiflat space is also $\epsilon(n)'$-almost flat, by Theorem 2.3.16 we can construct a new group Γ', and a nilpotent Lie group N' such that S is orbifold-diffeomorphic to N'/Γ'. The proof also gives that Γ' acts by affine transformations on N' and has a finite index normal subgroup $\Lambda' \lhd \Gamma'$, with $[\Gamma' : \Lambda'] \leq c(n)$, and Λ' acts on N' by translations.

Both N' and N constitute orbifold universal covers in the sense of [139, Chapter 13]. Thus, there is a diffeomorphism $f \colon N' \to N$ conjugating the action of Γ' to that of Γ. Denote the induced homomorphism by $f_* \colon \Gamma' \to \Gamma$.

By [8, Theorem 2] there is a finite index $\Lambda < \Gamma$, such that Λ acts by translations on N'. In particular, Λ is a co-compact subgroup of N. We still need to show that it is possible to choose Λ with a universal bound on the index $c(n)$, since [8] does not bound the index. We wish to translate the index bound for Λ' to one for Λ. Applying [8, Proposition 2] to the nilpotent group $f_*^{-1}(\Lambda')$ we get $f_*^{-1}(\Lambda') < \Lambda$. Thus, $[\Gamma : \Lambda] \leq c(n)$.

$\square$

In the previous section we defined generators for groups. For lattices in nilpotent Lie groups there are special classes of generators that we introduce here.

Definition 2.3.20. Let N be a simply connected nilpotent Lie group. A co-compact discrete subgroup $\Lambda < N$ is called a lattice.

Definition 2.3.21. Let N be a n-dimensional simply connected nilpotent Lie group and $\Lambda < N$ a lattice. Then a set $\{\gamma_1, \ldots, \gamma_n\} \subset \Lambda$ is called a triangular basis for Λ if the following properties hold.

- $\gamma_1, \ldots, \gamma_n$ generate Λ.

- $\gamma_i = e^{X_i}$, and X_i form a vector space basis for the Lie algebra $\mathfrak{n} \cong \mathbb{R}^n$ of N.

- For $i < j$ we have $[\gamma_i, \gamma_j] \in \langle \gamma_1, \ldots, \gamma_{i-1} \rangle$.

- For $i < j$ $[X_i, X_j] \in \mathrm{span}(X_1, \ldots, X_{i-1})$.

Here, e^X is the exponential map. In the case of $N = \mathbb{R}^n$ the previous definition reduces to the standard definition of a triangular basis for a Lattice. An immediate consequence of the previous definition is that the group N can be given co-ordinates by $(t_1, \ldots t_n) \to e^{t_1 X_1} \cdots e^{t_n X_n}$. This will be used in the final section of this chapter.

Lemma 2.3.22. *Let $S = N/\Gamma$ be an $\epsilon(n)$-quasiflat manifold, and $\Lambda = \Gamma \cap N$. For any $p \in N$ we can generate Γ and Λ by elements γ with $|\gamma|_p \le 16c(n)\epsilon(n)$.*

Proof: Consider the action of $\Lambda = \Gamma \cap N$. Then N/Λ is a Riemannian manifold. Further by Lemma 2.3.19, the index of Λ in Γ is at most $c(n)$, so there is a m-fold map $\pi \colon N/\Lambda \to N/\Gamma$ with $m \le c(n)$. Since Diam $(N/\Gamma) < \epsilon(n)$ and since N is connected, we have Diam $(N/\Lambda) < 2c(n)\epsilon(n)$.

Let $\Gamma' = \langle g \in \Gamma \,|\, |g|_p \leq 16\epsilon(n)c(n) \rangle$, and $\Lambda' = \langle g \in \Lambda \,|\, |g|_p \leq 16c(n)\epsilon(n) \rangle$. By Lemma 2.3.8 the space N/Γ' is isometric to N/Γ, and thus $\Gamma = \Gamma'$. Similarly, we can show $\Lambda = \Lambda'$. This completes the proof.

$\square$

The main conclusion of the following statement is the existence of a basis $\gamma_1, \ldots, \gamma_n$ with several technical properties. The latter part of the statement concludes the existence of associated subgroups Λ_k, L_k and scalars l_k with certain geometric properties. These parameters l_k are the "collapsing scales", and part of the conclusion is that they are independent of the chosen base point in a specific sense.

Lemma 2.3.23. *(Stratification lemma) Fix arbitrary integers $n \geq 1$ and $l > 4$, and denote $L = 2l^n$. For every $\epsilon(n)$-quasiflat orbifold $S = N/\Gamma$ there exists following objects with certain desired properties described below.*

1. *A triangular basis $\gamma_1, \ldots, \gamma_n$ for Λ.*

2. *A natural number s ("the number of collapsing scales").*

3. *For every $k = 1, \ldots, s$, subsets $I_k, J_k \subset \{1, \ldots, n\}$.*

4. *Subgroups $\Lambda_k = \langle \gamma_i | \gamma_i \in J_k \rangle \subset N$ corresponding to J_k, for $k = 1, \ldots, s$.*

5. *Connected normal subgroups $L_k \triangleleft N$ corresponding to Λ_k for $k = 1, \ldots, s$.*

6. *Scalars $l_k = 2|\gamma_{i_k}|_e > 0$, where $i_k = \max I_k$ ("collapsing scales") for each $k = 1, \ldots, s$.*

The basis γ_i and the groups $I_k \subset \{1, \ldots, n\}$ for $k = 1, \ldots, s$ satisfy the following.

- *$|\gamma_i|_e < |\gamma_j|_e$ for $i < j$.*

- *For all k and $\gamma_i, \gamma_j \in I_k$: $|\gamma_i|_e \leq L|\gamma_j|_e$.*

- *For all $\gamma_i \in I_s$ and $\gamma_j \in I_t$ and $s < t$: $l|\gamma_i|_e \leq |\gamma_j|_e$.*

We can set $J_k = \cup_{i=1}^{k} I_k$. The subgroups Λ_k are normal in Γ, and $\Lambda_k < L_k$ is a co-compact lattice in L_k. Further, the subgroups L_k are invariant with respect to the action of the image of the holonomy $h \colon \Gamma \to Aut(N)$.

Finally, the scalars l_k satisfy for every $p \in N$ and every $k = 1, \ldots, s$ that

$$\Lambda_k = \langle \lambda \in \Lambda | |\lambda|_p < l_k \rangle$$

and that

$$|\lambda|_p > l \cdot l_k / 4$$

for every $\lambda \in \Lambda \setminus \Lambda_k$. Also, for any non-trivial $\lambda \in \Lambda$ we have $|\lambda|_p > l_1/(2L)$.

Remark: The properties of l_k are needed to be able to apply Lemma 2.3.13.

Proof: The basis is constructed similarly to [26]. Let γ_1 minimize $|\cdot|_e = |\cdot|$ in Λ. Set $G_0 = \{e\}$ to be the trivial group. Define the subgroup G_1 generated by γ_1. Proceed to choose $\gamma_2 \in \Lambda \setminus G_1$ as the shortest with respect to $|\cdot|_e$. Define $G_2 = \langle \gamma_1, \gamma_2 \rangle$. We proceed inductively to define $\gamma_1, \ldots, \gamma_n$. The element γ_i is the shortest element with respect to $|\cdot|_e$ in $\Lambda \setminus G_{i-1}$ and $G_i = \langle \gamma_1, \ldots, \gamma_i \rangle$. A priori, this process could last more than n steps, but by an argument below the number

of γ_i agrees with the dimension n. Thus, we abuse notation slightly by using the same index.

Take any $a, b \in N$ such that $|a|_e, |b|_e \leq 16c(n)\epsilon(n)$. We have

$$|[a, b]|_e \leq 16C'(n)c(n)\epsilon(n) \min\{|a|_e, |b|_e\}. \tag{2.3.24}$$

This estimate follows either from curvature estimates such as in [26], or directly from the bounds on the structural constants on N. Choose $\epsilon(n) < \frac{1}{10^2 c(n)C'(n)}$, so that $|[a, b]|_e < \min\{|a|_e, |b|_e\}$.

Because of Lemma 2.3.22 we can choose generators α_i for the translational subgroup $\Lambda < \Gamma$ such that $|\alpha_i|_e < 16c(n)\epsilon(n)$. Moreover, by construction we get then that $|\gamma_i| < 16c(n)\epsilon(n)$. By estimate (2.3.24), we have $|[\alpha_i, \gamma_i]|_e < |\gamma_i|_e$, and thus $[\alpha_i, \gamma_i] \in G_{i-1}$ by construction. Since α_i generate Λ, we get for any $\lambda \in \Lambda$ that $[\lambda, \gamma_i] \in G_{i-1}$. In particular, for $i < j$ we have $[\gamma_i, \gamma_j] \in G_{i-1}$.

We define a map $f: (a_1, \ldots, a_n) \to \gamma_1^{a_1} \ldots \gamma_n^{a_n}$. By induction, and the product relation below, we can show that the map is injective. Using the commutator relations from above and reordering terms, we can show that f is bijective onto Λ. Let $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n)$. We can define polynomials p_j by the defining relation

$$\gamma_1^{a_1} \ldots \gamma_n^{a_n} \times \gamma_1^{b_1} \ldots \gamma_n^{b_n} = \gamma_1^{a_1+b_1+p_1(a,b)} \ldots \gamma_n^{a_n+b_n+p_n(a,b)}.$$

The polynomials are defined by appropriately applying commutator relations. Also, p_i depends only on a_j, b_j for $j < i$. This defines a product on $\mathbb{Z}^n$, which can be extended to $\mathbb{R}^n$ by the previous relation. This defines a nilpotent Lie group N' with a co-compact lattice $\mathbb{Z}^n$. The lattice has a standard basis $e_i = f^{-1}(\gamma_i) \in \mathbb{Z}^n$,

and $e_i = e^{X_i}$, where $X_i' = (0, \ldots, 1, \ldots, 0) \in T_0 \mathbb{R}^n$ with the 1 in the i'th position. By a Theorem of Malcev (see [102] and [31]) we can extend the homomorphism $f: \mathbb{Z}^n \to \Lambda$ to a bijective homomorphism $\overline{f}: N' \to N$. Since the map is bijective and since X_i' form a basis at $T_e N$, then $d\overline{f}(X_i') = X_i$ will form a basis for $T_e N$. Further $\gamma_i = e^{X_i}$, and $[X_i, X_j] = d\overline{f}([X_i', X_j'])$ it is direct to verify that $[X_i, X_j] \in \text{span}(X_1, \ldots, X_{i-1})$ for $j > i$. In particular the basis is triangular. Moreover, the number of basis elements has to agree the dimension of the space.

Next, we define the sets I_j. Define $i_0 = 1$. Let i_1 be the smallest index such that $|\gamma_{i_1+1}| > l|\gamma_{i_1}|$, or if there is none $i_1 = n$. Continue inductively defining i_k to be the smallest index bigger than i_{k-1} such that $|\gamma_{i_k+1}| > l|\gamma_{i_k}|$, or if there is no such index $i_k = n$ and we stop. Define s such that $i_s = n$, and $I_k = \{\gamma_i | i_{k-1} < i \leq i_k\}$. The properties in the statement are easy to verify. The k most collapsed directions are collected into $J_k = \bigcup_{i=1}^k I_i$, and we define $\Lambda_k = \langle \gamma_i | \gamma_i \in J_k \rangle$. Define also $\Lambda_0 = \{e\}$

By choosing $\epsilon(n) < \frac{1}{10^2 c(n) C'(n) L}$ and applying (2.3.24), we can prove that, if $\gamma_i \in I_s, \gamma_j \in I_t$ and $s < t$, then $[\gamma_i, \gamma_j] \in \Lambda_{s-1}$. We remark that by virtue of the construction, if $\gamma_i \in \Lambda_k$, and $\gamma \in \Lambda$ such that $|\gamma| < l|\gamma_i|$, then also $\gamma \in \Lambda_k$.

Since γ_i form a basis, we can show that Λ_k is normal in Λ. Next, we want to show that it is also normal in Γ. Take an element $\sigma \in \Gamma$ with non-trivial holonomy, and a $\gamma_i \in \Lambda_k$ from the basis. Further represent $\sigma = (t, A)$, where $A \in \text{Aut}(N)$ is in the holonomy and $t \in N$. By left-multiplying with an element in Λ, we can assume $|t|_e = |\sigma|_e \leq 16c(n)\epsilon(n)$. This follows from the diameter bound for N/Λ in Lemma 2.3.22. Then by normality of Λ, $\sigma\gamma_i\sigma^{-1} = t(A\gamma_i)t^{-1} \in \Lambda$. Since A is an isometry, $|A\gamma_i| = |\gamma_i|$. This combined with the commutator estimate (2.3.24) gives $d(t(A\gamma_i)t^{-1}, A\gamma_i) \leq \frac{1}{10}|\gamma_i|$, and thus $|t(A\gamma_i)t^{-1}| \leq (1 + \frac{1}{10})|\gamma_i|$. By the iterative definition of Λ_k, we must have $t(A\gamma_i)t^{-1} \in \Lambda_k$. Note that we are assuming $l > 2$.

The connected Lie subgroups L_k can be generated by $X_1, \ldots, X_{i_k}$, and normality in Λ follows since the basis is triangular. It is also easy to see that L_k is normal in N. Further, co-compactness of Λ_k follows by considering the map $(t_1, \ldots, t_{i+k}) \to e^{t_1 X_1} \cdots e^{t_{i_k} X_k}$. By the argument in the previous paragraph, for any element $\sigma = (t, A) \in \Gamma$ with $|\sigma|_e < 16\epsilon(n)c(n)$ and $\gamma \in \Lambda_k$, we have $tA\gamma t^{-1} \in \Lambda_k \in L_k$. Since L_k is normal in N we have $A\gamma \in L_k$. From this, we can conclude that $AL_k = L_k$, i.e. that L_k are invariant under the holonomy action of Γ.

Finally, define $l_k = 2|\gamma_{i_k}|$. Fix $p \in N$. We will show that $\Lambda_k = \langle \lambda \in \Lambda | d(\lambda p, p) < l_k \rangle$ and $|\lambda|_p = d(\lambda p, p) > l/4l_k$ for every $\lambda \in \Lambda \setminus \Lambda_k$. First of all, we can assume that $d(p, e) < 16c(n)\epsilon(n)$. This follows again from the diameter bound for N/Λ in Lemma 2.3.22. By assumption for any $\gamma_i \in \Lambda_k$ we have $|\gamma_i|_e \leq |\gamma_{i_k}|_e < \frac{1}{2}l_k$. Thus, by the commutator estimate (2.3.24),

$$d(\gamma_i p, p) = d(p^{-1}\gamma_i p, e) \leq d(p^{-1}\gamma_i p, \gamma_i) + d(\gamma_i, e) \leq \left(1 + \frac{1}{10}\right)|\gamma_{i_k}|_e < l_k.$$

Thus, $\gamma_i \in \langle \lambda \in \Lambda | d(\lambda p, p) < l_k \rangle$, and further by varying γ_i and generation we get $\Lambda_k < \langle \lambda \in \Lambda | d(\lambda p, p) < l_k \rangle$. Next if $\gamma \notin \Lambda_k$, then $|\gamma|_e = d(\gamma, e) > l/2l_k$ by the construction of Λ_k. Simple estimates give

$$d(\gamma p, p) = d(p^{-1}\gamma p, e) \geq d(\gamma, e) - d(p^{-1}\gamma p, \gamma) \geq (1 - \frac{1}{10})|\gamma|_e \geq l/4l_k.$$

In particular, for every element such that $d(\lambda p, p) < l_k < l/4l_k$ we must have $\lambda \in \Lambda_k$. Thus, $\Lambda_k = \langle \lambda \in \Lambda | d(\lambda p, p) < l_k \rangle$ and for any $\gamma \notin \Lambda_k$ we have $|\lambda|_p > l/4l_k$.

The final conclusion, that $d(\lambda p, p) > l_1/(2L)$ for all $\lambda \in \Lambda$, follows from $|\lambda|_e > l_1/L$ which follows from the construction of γ_i.

$\square$

It follows from the previous lemma that the scales l_k and subgroups Λ_k, L_k are essentially independent of the chosen base point. Further, they correspond to "local groups" at scales l_k.

Definition 2.3.25. Let $S = N/\Gamma$ be an $\epsilon(n)$-quasiflat orbifold. If $l = 400c(n)$ and $L = 2l^n$, then call a basis $\gamma_1, \ldots, \gamma_n$ satisfying the previous lemma a short basis for Λ. The grouping $I_1, \ldots, I_s$ is called canonical grouping and the number s is called the number of collapsing scales. If such a basis exists, we say that S has s collapsing scales. For $k = 1, \ldots, s$, the sets J_k are called the k'th most collapsed scales.

Definition 2.3.26. Let S be a complete Riemannian orbifold and $L = V(S)$ an orbivector bundle over S. Then L is called a (k-dimensional) *locally flat (Riemannian) orbivector bundle* if the orbivector bundle $V(S)$ admits a bundle-metric and a metric connection such that it is locally flat (and such that the fibers have dimension k). In the case where V and S are complete Riemannian manifolds we call V a locally flat Riemannian vector bundle over S.

A bundle metric is a metric on the fibers of S which is parallel with respect to the connection. For the notion of an orbivector bundle one may consult [89] and the final section of this chapter. The notions of a bundle metric and metric connection are naturally extended from vector bundles to orbivector bundles by

considering local charts. Note that we permit the case when $S = \{\text{pt}\}$ is a single point (i.e. a zero-dimensional orbifold), and $L = \mathbb{R}^k/\Gamma$, where $\Gamma < O(k)$ is a finite group.

Because the monodromy action[6] is by isometries, and the bundle is locally trivial, there is a canonical Riemannian metric and distance function on L (and not just on the fibers). To understand the geometry of L we are reduced to understanding the geometry of S and that of the monodromy action. These metrics are somewhat easier to visualize in the case where S is a compact flat manifold, and V is a locally flat vector bundle over such a base. In this case L is given by a quotient of Euclidean space by an action of a free, discrete and isometric group action, which may fail to be co-compact.

In our case, all the orbifolds $L = V(S)$ will arise as group quotients. By assumption, S will be $\epsilon(n)$-quasiflat. Thus, S has an orbifold universal cover N, which is a simply connected nilpotent lie group. Further, S is given by the quotient N/Γ, where Γ is the orbifold fundamental group, which acts discretely, co-compactly and by isometries on N. The orbivector bundle L has a pull-back vector bundle L' on N. Since N is simply connected, and L was assumed locally flat, we have a flat pull-back connection on L' making it trivial. In particular, we have, by choosing an orthonormal frame on some fiber, a natural trivialization $L' = N \times \mathbb{R}^k$, where k is the dimension of the fiber. The pull-back metric on L' is simply a product metric, and the distance is given by the product structure. From this perspective, the distance on L is given by a quotient metric on L'/Γ, where Γ acts on the $\mathbb{R}^k$ factor via a monodromy action. Note, all these vector bundles are quotients and will be studied as such. However, they are not *compact*, because the

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[6]Monodromy may also be defined for orbifold. A similar discussion is used to prove that flat orbifolds are good in [120].

action of Γ on the space $N \times \mathbb{R}^k$ is not co-compact.

The monodromy action is given by a homomorphism $\Gamma \to O(n)$, where Γ corresponds to the orbifold fundamental group. The rest of this section collects some necessary facts about these group representations. Note that in our case Γ is "virtually" nilpotent, and its action on $\mathbb{R}^n$ will turn out to be "virtually" abelian. By "virtually" we mean that there is a finite index subgroup with the desired property, and that the index can be controlled in terms of the dimension n.

Definition 2.3.27. Let H be an abelian subgroup of $O(n)$. A decomposition $\mathbb{R}^n = \bigoplus V_i$ is called the *canonical decomposition* and the subspaces $V_0, \ldots, V_k$ are called *canonical invariant subspaces* if they satisfy the following properties.

- Each subspace has positive dimension, except possibly V_0.

- For every $h \in H$, and every j we have $hV_j = V_j$.

- There is one V_j such that for all $h \in H$ we have $hv = v$ for all $v \in V_j$ (i.e. each $h \in H$ acts trivially on this subspace). We will always denote this subset V_0 possibly allowing $V_0 = \{0\}$

- On each other subspace V_j either one of the following holds.

  1. For every $h \in H$ and every $v \in V_j$ either $hv = v$ or $hv = -v$

  2. V_j is even dimensional and we can associate isometrically complex coordinates $(z_1, \ldots, z_n)$, such that for any $h \in H$ the action is given by $h((z_1, \ldots, z_n)) = \left(e^{\theta_j(h)i} z_1, \ldots, e^{\theta_j(h)i} z_n\right)$ for some $\theta_j(h) > 0$.

- The subspaces are maximal with respect to these properties and pairwise orthogonal.

If $A \in O(n)$ is an isometry, the canonical decomposition associated to it is the one associated to the cyclic subgroup generated by A.

Every abelian isometric action has a unique canonical decomposition. For an individual matrix, this should be thought of as a decomposition corresponding to the factorization of its minimal polynomial. Using this terminology we prove a minor modification of the Jordan lemma.

Lemma 2.3.28. *(Modified Jordan Theorem) There is a constant $c''(n)$ depending on the dimension with the following property. Let Γ be a group with a nilpotent normal subgroup Λ with index $[\Gamma : \Lambda] < \infty$ and $h \colon \Gamma \to O(n)$ a homomorphism. Then $h(\Gamma)$ has a normal abelian subgroup of index at most $c''(n)$.*

Proof: Let $L = \overline{h(\Gamma)}$ be the closure of the image of h. The subgroup L is Lie subgroup of $O(n)$. It is either finite or has non-trivial identity component T. In the first case, we may proceed as in the standard Jordan Theorem [138]. In the latter case, since Γ has a finite index nilpotent subgroup, $T \subset O(n)$ must be a compact nilpotent group. This is only possible if it is abelian, i.e. a torus. The torus T corresponds to a decomposition of $\mathbb{R}^n$ into canonical invariant subspaces V_j. Since T is a continuous group, except possibly for one V_0, every V_j for $j = 1, \ldots, s$ will be even-dimensional, and the action will correspond to a Hopf-action $\theta(z_1, \ldots, z_k) \to (e^{i\theta} z_1, \ldots, e^{i\theta} z_k)$. On V_0 the action is trivial.

Since T is normal in L, the decomposition to V_j's corresponds to a homomorphism

$$\phi = (\phi_1, \ldots, \phi_k) \colon \ L \to \prod_{j=1}^{s} O(V_j).$$

Also, the action of L on each subspace conjugates the Hopf-action on each V_i to itself (which agrees with the action of T). The infinitesimal generator in $\mathfrak{o}(V_j)$

corresponding to the Hopf-action is J which maps $J(z_1, \ldots, z_k) = (iz_1, \ldots, iz_k)$, if $\dim(V_j) = 2k$. In particular for any $g \in \phi(L)$ we have $gJg^{-1} = J$ or $gJg^{-1} = -J$ (because it preserves the action), which means that g either preserves the complex structure, or it composed with the conjugate map does so. In other words, g is either symplectic or a product of a symplectic matrix with a matrix B giving the conjugation on V_j. Thus, $\phi_j(L) \subset U(V_j) \cup \overline{U(V_j)}$, where $U(V_j)$ is the symplectic group of isometries and is isomorphic to the unitary group of k by k matrices, and $\overline{U(V_j)}$ is obtained by composing such a matrix with B.

Consider $\phi(L) \cap \prod_j (SU(V_j) \cup \overline{SU(V_j)}) = K$, where $SU(V)$ and $\overline{SU(N)}$ are those matrices in $U(V_j)$ with unit determinant (using the k by k complex co-coordinates), or their product with B. This group must be finite because every left coset of T contains a finite number of elements in $\prod_j SU(V_j)$. Here, we used that the group T acts on V_j via a Hopf-action $\theta(z_1, \ldots, z_n) \to (e^{i\theta}z_1, \ldots, e^{i\theta}z_n)$, which has determinant $e^{i\dim(V_j)\theta}$. Also, $e^{i\dim(V_j)\theta} = 1$ only when $\theta = 2\pi n/\dim(V_j)$. Since K is finite, conjugation by T will leave it fixed, and so any element of T commutes with any element in K. Further, $\phi(L) = KT$ by the previous discussion.

By the standard Jordan theorem, the group K has a finite index abelian normal subgroup $H \triangleleft K$ of index at most $d(n) \leq O((n+1)!)$. Define the abelian group $G = HT$, which is generated by H and T. We have

$$[\phi(L) : G] = [KT : HT] \leq [K : H].$$

Since we have an effective bound for the index $[K : H] \leq d(n)$, we get a bound for $[\phi(L) : G] \leq d(n)$. It is also easy to see that G is normal in $\phi(L)$.

$\square$

Remark: The optimal bound for the index in the classical Jordan theorem is $(n+1)!$, which is valid for $n \geq 71$ [44]. Thus, from the previous argument we get the optimal bound $[\Gamma : \Lambda] \leq c''(n) \leq (n+1)!$ for $n \geq 71$. Below, we will set $c(n)$ to be the larger of $c(n)''$ in Lemma 2.3.28 and $c(n)$ in Theorem 2.3.16.

2.4 Construction of embeddings

The proof will proceed by increasing generalities. First we discuss the case of flat manifolds, then the case of flat orbifolds, and finally the case of quasiflat orbifolds. At the end, we use Theorems 2.1.3 and 2.1.4 to obtain the proof for Theorem 1.2.22.

2.4.1 Flat Manifolds

In this section we construct embeddings for flat manifolds by proving the following.

Theorem 2.4.1. *Every connected complete flat manifold M of dimension n admits a bi-Lipschitz embedding $f \colon M \to \mathbb{R}^N$ with distortion less than $D(n)$ and dimension of the image $N \leq N(n)$.*

We will prove this theorem by induction. The somewhat complicated induction argument is easier to understand through examples for which we refer to Section 2.2. While it is true that the proof could be directly done for orbifolds, the author believes it is easier to start with the manifold context and later highlight the small differences with the orbifold case.

The goal is to use embeddings at different scales and reveal ever simpler vector bundles at smaller scales. Simplicity is measured in the total dimensionality of the space or the dimensionality of the base space of a vector bundle. This is also a frequent idea of collapsing theory, and as such the proof can be seen as a refinement of collapsing theory to a family of explicit spaces. To avoid lengthy phrases, we use $V - (n, d)$ to abbreviate the induction statement, where $d \geq 0$ and $n > 0$. The case $V - (n, 0)$ covers *compact* cases of the theorem, and $V - (n, d)$ for $d > 0$ covers the *non-compact* cases, which are vector bundles. Bieberbach's theorem 2.3.14 can be used to show that any connected complete flat manifold is covered by either of these two cases. The notation $V - (n, 0)$ is justified by imagining compact manifolds as 0-dimensional vector bundles over themselves.

Next, we define these statements more carefully. The case $d = 0$ is treated in a special way.

- $V - (n, 0)$: There exist constants $L(n, 0)$ and $N(n, 0)$ such that the following holds. Assume M is an arbitrary n-dimensional connected compact flat manifold. Then, there exists a bi-Lipschitz embedding $f \colon M \to \mathbb{R}^N$ with distortion $L \leq L(n, 0)$ and target dimension $N \leq N(n, 0)$ depending only on n.

- $V - (n, d)$: There exist constants $L(n, d)$ and $N(n, d)$ such that the following holds. Assume V is an arbitrary d-dimensional connected locally flat Riemannian vector bundle over a compact flat manifold M of dimension n. Then, there exists a bi-Lipschitz embedding $f \colon V \to \mathbb{R}^N$ with distortion $L \leq L(n, d)$ and target dimension $N \leq N(n, d)$ depending only on n, d.

In the proof below we will perform a double induction to prove $V - (n, d)$,

which is based on reducing the embedding problem for $V - (n, d)$ to embedding instances of $V - (n - k, k + d)$ for some $k > 0$ or of $V - (a, b)$ with $a + b < n + d$. For $V - (n, 0)$ the reduction is similar to that of a lens space, and for $V - (n, d)$ the reduction is analogous to that for holonomy-bundles (see Section 2.2).

The main steps of these reductions are choosing a length scale $\delta > 0$, applying a stratification of a lattice from Lemma 2.3.23, followed by an application of the local group Lemma 2.3.13 to expose the structure of a local group at the scale δ, and then applying Lemma 2.3.8 to describe the metric structure at those scales. The metric structure at a small but definite scale is always that of a vector bundle. The base of such a vector bundle is an embedded manifold of smaller dimension. This embedded manifold is found by finding submanifolds $\mathbb{R}^k$ in the universal cover - that is $\mathbb{R}^n$ - which remain invariant under the action of a local group. Here, the averaging from Lemma 2.3.11 plays a crucial role. Once the embeddings are constructed at scale δ, using the induction hypothesis, the embedding of the space is obtained by using Lemma 2.3.5.

Proof or Theorem 2.4.1: We first give a description of the induction argument, and then move onto defining the relevant notation and proving the induction claims.

Summary of induction argument: We need to embed all complete connected flat manifolds. By Biberbach's theorem 2.3.14 every complete connected flat manifold can be represented as a quotient by a discrete group of isometries. In particular, any such manifold is either isometric to $\mathbb{R}^n$, a quotient of $\mathbb{R}^n$ by a co-compact group action or a locally flat vector bundle over such a base. In the first case, the embedding is given by the identity mapping. Otherwise, either the space can

be represented as $M = \mathbb{R}^n / \Gamma$, or $V = \mathbb{R}^n \times \mathbb{R}^d / \Gamma$, where Γ is some discrete Lie group which acts co-compactly, properly discontinuously and by isometries on $\mathbb{R}^n$. Additionally, for V, the action of Γ leaves $\mathbb{R}^n \times \{0\}$ invariant, and acts by orthogonal tranformations on the fibers $\mathbb{R}^d$. In the latter case, the action is *not* co-compact on all of $\mathbb{R}^n \times \mathbb{R}^d$, and the quotient will be a non-compact space. The case $V - (n, 0)$ covers the problem of embedding M, and $V - (n, d)$ covers the problem of embedding V.

Thus, we need to show $V - (n, d)$ for $n \geq 1, d \geq 0$. First, we induct on $n + d$. If $n + d = 1$, then we only have the case $V - (1, 0)$, which is the base case for our induction.

Next, assume the statement has been shown for $n + d \leq m$, and we prove it for $V - (n, d)$ with $n + d = m + 1$, that is $V - (k, m + 1 - k)$ for $k = 1, \ldots, m + 1$. We show this by induction on k. The base case is $V - (1, m)$ which is reduced to $V - (a, b)$ with $a + b \leq m$. Note $S = 1$ in this case.

Assume now that we have shown $V - (k, m + 1 - k)$ for all $k = 1, \ldots, k' - 1$ with $k' < m + 1$ and show it for $k = k'$. If $k' < m + 1$, then the third reduction below reduces the embedding problem to instances of $V - (a, b)$ with either $a < k'$ or $a + b \leq m$. In both cases we have reduced the embedding problem to cases we have assumed to be true. For $k' = m + 1$ we want to prove $V - (m + 1, 0)$, which is reduced to embedding instance of $V - (a, b)$ with $a < m + 1$. By the induction principle, we can assume these are already embedded, and thus the statement follows.

Defining variables: We will use Γ to denote the group acting on $\mathbb{R}^n \times \mathbb{R}^d$ and $\mathbb{R}^n$ as described above. By Biberbach's theorem, since the action of Γ on $\mathbb{R}^n$ is

co-compact, there is a finite index lattice $\Lambda = \Gamma \cap \mathbb{R}^n$ acting by translations on $\mathbb{R}^n$ with index $[\Gamma : \Lambda] \leq c(n)$. Denote by 0 the identity element of $\mathbb{R}^n$ and use the subadditive norm $|\gamma|_0 = |\gamma| = d(\gamma 0, \gamma)$ on Γ. Denote by e the identity element of Γ. Denote by $D = \text{Diam}(M) = \text{Diam}(\mathbb{R}^n / \Gamma)$. Apply Lemma 2.3.23 to choose the following objects. Choose a short basis $\gamma_1, \ldots, \gamma_n$ for Λ, and the canonical grouping $I_1, \ldots, I_S$ with parameters $l = 400c(n)$ and $L = 2l^n$ (see Definition 2.3.25). Let S be the number of collapsed scaled. By J_k for $k = 1, \ldots, S$ we denote the k'th most collapsed scales. By the proof for Lemma 2.3.23, these correspond to normal subgroups $\Lambda_k < \Lambda$, which are normal in Γ. Also, they correspond to connected normal Lie-subgroups $L_k < \mathbb{R}^n$ which are invariant under the holonomy action of Γ, and such that $\Lambda_k < L_k$ are co-compact. Denote by l_k the collapsing scales given by Lemma 2.3.23.

Points in $\mathbb{R}^n$ are denoted by lower case letters such as p and points in the quotient space by $[p]$. Cosets with respect to local groups will be denoted by $[p]_r$. We will now proceed by induction on n and $n + d$ to show $V - (n, d)$ for all n.

Base case $V - (1, 0)$: In this case the flat manifold is one-dimensional and isometric to S^1. Such a manifold is trivial to embed.

Case $V - (n, 0)$ when $S = 1$: From the definition of $l_1 = 2|\gamma_n|_0$ in Lemma 2.3.23, and Lemma 2.5.4 we obtain $\text{Diam}(M) \leq C(n)l_1$. Let $p \in \mathbb{R}^n$ be arbitrary. Set $\delta = l_1 / (320c(n)L)$. Define the group $\Gamma_\delta = \langle g \in \Gamma | d(gp, p) < 8\delta \rangle$. By Lemma 2.3.8 for any $[p] \in M$ we have $B_{[p]}(\delta) \subset M$ is isometric to $B_{[p]_\delta}(\delta) \subset \mathbb{R}^n / \Gamma_\delta(p)$. We next describe the structure of the group $\Gamma_\delta(p)$.

Apply Lemma 2.3.13 with $l = 8\delta$. When we define $\Lambda_0 = \langle \lambda \in \Lambda | d(\lambda p, p) < 8\delta \rangle = \{e\}$, and observe that by Lemma 2.3.23 for any $\lambda \in \Lambda$ we have $|\lambda|_e >$

$l_1/(2L) \geq 80c(n)\delta$, we obtain a short exact sequence

$$0 \to 0 \to \Gamma_\delta \to H_0 \to 0.$$

The group H_0 is finite and thus also Γ_δ is finite. Since M is a manifold, Γ_δ acts freely on $\mathbb{R}^n$. Applying Lemma 2.3.11, we get that, if Γ_δ were non-trivial, then it would have a fixed point. Since this would contradict that the action of Γ_δ is free, we must have Γ_δ is trivial. Thus, $B_{[p]_\delta}(\delta) \subset \mathbb{R}^n/\Gamma_\delta(p)$ is isometric to a Euclidean ball of the same radius $B_0(\delta) \subset \mathbb{R}^n$. Thus, each ball $B_{[p]}(\delta)$ is isometric to a subset of $\mathbb{R}^n$, and obviously admits a bi-Lipschitz embedding to $\mathbb{R}^n$. Also, Diam $(M)/\delta \leq 320C(n)c(n)L$. Since p was arbitrary, Lemma 2.3.5 implies that M also possesses an embedding.

Reduction $V - (n, 0)$ to $V - (n - k, k)$ for some $n > k \geq 1$ when $S > 1$: The gain here is that the dimension of the new base space $n - k$ is smaller than n. Consider $\delta = \max\left(\frac{l_S}{200Lc(n)}, 2l_{S-1}\right)$. Note that Diam $(M) < C(n)l_S \leq 200Lc(n)C(n)\delta$ by Lemma 2.5.4. Thus, similarly to the previous case, it is sufficient by Lemma 2.3.5 to embed each ball $B_{[p]}(\delta)$. Next fix $[p] \in M$, and its representative $p \in \mathbb{R}^n$. We will show that $B_{[p]}(\delta)$ is isometric to a subset of a vector bundle over a $(n - k)$-dimensional flat manifold for some k.

Define the local group $\Gamma_\delta = \langle g \in \Gamma | d(gp, p) < 8\delta \rangle$, and using Lemma 2.3.8 we get $B_{[p]}(\delta) \subset M$ is isometric to $B_{[p]_\delta}(\delta) \subset \mathbb{R}^n/\Gamma_\delta$. Define the subgroup $\Lambda_\delta = \langle \lambda \in \Lambda | d(\lambda p, p) < 8\delta \rangle$. By Lemma 2.3.23 for each $\lambda \notin \Lambda_{S-1}$ we have $d(\lambda p, p) > l \cdot l_{S-1}/4 > 80c(n)\delta$, and thus we can see that $\Lambda_\delta = \Lambda_{S-1}$. Further, by Lemma 2.3.13 applied with $l = 8\delta$ there is a finite group H_0 and a short exact sequence

$$0 \to \Lambda_{S-1} \to \Gamma_\delta \to H_0 \to 0.$$

Consider the connected Lie-subgroup L_{S-1} corresponding to Λ_{S-1} via Lemma 2.3.23. The subgroup L_{S-1} is invariant under Γ_δ and the holonomy. Thus, the action of Γ_δ descends to an action of H_0 on the orbit-space $E = \mathbb{R}^n/L_{S_1}$. Represent elements in E as orbits $[x]_{L_{S-1}}$ for $x \in \mathbb{R}^n$. Thus, by Lemma 2.3.11 there is an orbit $[o]_{L_{S-1}} \in E$ fixed by the action of H_0. Let $O = [o]_{L_{S-1}} \subset \mathbb{R}^n$ be the orbit. By construction O is invariant under the group action Γ_δ. Since O is invariant under Γ_δ we have $\mathbb{R}^n/\Gamma_\delta = V'$, where V' is a locally flat Riemannian vector bundle over O/Γ_δ of dimension $n - k$. Also, the action of Γ_δ is co-compact on O, because $\Lambda_{S-1} < \Gamma_\delta$. In particular $B_{[p]_\delta}(\delta)$ is isometric to a subset of a k-dimensional locally flat Riemannian vector bundle over a $n - k$-dimensional compact connected flat manifold. Thus, the ball $B_{[p]}(\delta)$ can be embedded by $V - (n - k, k)$.

Reduction $V - (n, d)$ to $V - (n - k, d + k)$ (if $S > 1$) and cases of the form $V - (a, b)$ with $a + b < n + d$: Denote by $V = \mathbb{R}^n \times \mathbb{R}^d/\Gamma$ the locally flat vector bundle over $M = \mathbb{R}^n \times \{0\}/\Gamma \subset V$. This reduction guarantees that either the dimension of the base space drops, or the total dimension drops. The proof is similar to the holonomy bundle example discussed in Section 2.2.

The space V is divided into sets defined by their distance to the zero-section M. Take the sets $T_0 = \{p \in V | d(p, M) \leq 2D\}$ and $T_k = \{p \in V | 2^{k+1}D \geq d(p, M) \geq 2^{k-1}D\}$ for $k \in \mathbb{N}$. Recall, $D = \text{Diam}(M)$. We first show that it is sufficient to construct an embedding f_k for each T_k of bounded dimension and distortion. Suppose such embeddings exist. First scale and translate f_k so that $0 \in \text{Im}(f_k)$ and f_k is 1-Lipschitz. Consider the sets $D_s = \bigcup_{k=0}^{\infty} T_{s+4k}$, and define $F_s(x) = f_{s+4k}(x)$

on T_{s+4k}. The resulting functions will be Lipschitz on their respective domains D_s. By an application of McShane extension Theorem 1.2.7, we can extend them to all of V. A bi-Lipschitz embedding will result from

$$G(x) = (r(x), F_0(x), F_1(x), F_2(x), F_3(x)),$$

where $r(x) = d(x, M)$. To see that this is an embedding, repeat the argument from Section 2.2. Next, construct embeddings f_i for each T_i.

First, we consider T_i for $i \geq 1$. Define $K = \{r(x) = 2^i D\}$. It is sufficient to find an embedding for the set K with it's induced metric. The metric space T_i is bi-Lipschitz equivalent to $[2^{i-1}, 2^{i+1}] \times K$, and we can use the product embedding from Lemma 2.3.1 to embed it. Therefore, we can focus on embedding K. If $d = 1, 2$, K will be compact and flat, and we can apply the case $V - (n + d - 1, 0)$ or the case $V - (n + d - 2, 0)$ to embed K. If $d > 2$, then K is not flat. But it is lower dimensional and has the structure of a sphere bundle. For each $q \in K$ we find a ball with radius proportional to $2^i D$ and which is bi-Lipschitz to a lower dimensional vector bundle. This combined with the doubling Lemma 2.3.5 gives an embedding for T_i.

For a real number t denote by $t S^{d-1}$ the $d - 1$-dimensional sphere of radius t. The space K can be represented as $K = \mathbb{R}^n \times D2^i S^{d-1}/\Gamma$, where Γ acts on S^{d-1} by isometries via the holonomy action $h \colon \Gamma \to O(d)$. Any point in K can be represented as an orbit $[(q, v)] \in K$ for an element $(q, v) \in \mathbb{R}^n \times D2^i S^{d-1}$. Fix $\delta = \frac{D2^i}{10^5 c(d)\sqrt{d}}$. We will show that $B_{[(q,v)]}(\delta) \subset K$ is bi-Lipschitz to a subset of locally flat vector bundle V of total dimension $m = n - 1$. This will be done by studying the local group $\Gamma_\delta = \langle g \in \Gamma | d(g(q, v), (q, v)) \leq 8\delta \rangle$, and observing by Lemma 2.3.8 that $B_{[q]}(\delta)$ is isometric to $B_{[(q,v)]_\delta}(\delta) \subset \mathbb{R}^n \times D2^i S^{d-1}/\Gamma_\delta$ and describing in detail

the action of Γ_δ.

By the modified Jordan Theorem 2.3.28, the group $h(\Gamma)$ has a normal abelian subgroup L of index at most $c(d)$, and $h^{-1}(L) = \Lambda_L$ is also normal and has finite index at most $c(d)$. Decompose $\mathbb{R}^d$ into canonical invariant subspaces V_j corresponding to L. By uniqueness, the subspaces V_j are invariant under the action of Γ. Decompose v into its components $v_j \in V_j$. Consider the induced norm $|| \cdot ||$ on $\mathbb{R}^d$. There is an index j such that $||v_j|| \geq \frac{2^i D}{3\sqrt{d}}$. To reduce clutter, re-index so that $j = 1$ and denote $\xi_1 = D2^i v_1/||v_1||$. Denote by $h_1 \colon \Gamma \to O(V_1)$ the action of Γ on V_1. Denote by $d_{S^{d-1}}$ the metric on the sphere $2^i D S^{d-1}$. Then

$$d_{S^{d-1}}(v, \xi_1) \leq (2^i D) \cos^{-1}\left(\frac{1}{3\sqrt{d}}\right) \leq 2^i D\left(\pi/2 - \frac{1}{3\sqrt{d}}\right).$$

For each generator g of Γ_δ we have $d(g(q, v), (q, v)) \leq 8\delta$. Then also

$$d_{S^{d-1}}(g\xi_1, \xi_1) \leq 24\delta\sqrt{d}. \tag{2.4.2}$$

Denote by $\Lambda_0 = \Gamma_\delta \cap \Lambda_L$. Clearly $[\Gamma_\delta : \Lambda_0] \leq c(d)$. Also, define $\Lambda_\delta = \langle g \in \Lambda_L | d_{S^{d-1}}(g\xi_1, \xi_1) \leq 24\delta\sqrt{d}\rangle$.

Either the action of L, and of Λ_L, on the sub-space V_1 will consist of reflections and identity transformations, or it will be a Hopf-type action. Consider the first case. We have $24c(d)\delta\sqrt{d} < D2^i\pi/4$, and thus the action of Λ_δ on V_1 is trivial. Consider the semi-norm $|g|'_{\xi_1} = d_{S^{d-1}}(g\xi_1, \xi_1)$, and the action of the groups $h_1(\Lambda_L)$ and $h_1(\Gamma)$ on $O(V_1)$ with the parameter $l = 24\delta\sqrt{d}$. Applying Lemma 2.3.13, and using (2.4.2) we get a short exact sequence

$$0 \to h_1(\Lambda_\delta) \to h_1(\Gamma_\delta) \to H_0 \to 0,$$

where H_0 is a finite group of order at most $c(d)$.

Since the action of Λ_δ on V_1 is trivial, we get that $h_1(\Gamma_\delta)$ is finite. By equation (2.4.2) and the product bound of Lemma 2.3.13, we get for any $g \in h_1(\Gamma_\delta)$ that $d_{S^{d-1}}(g\xi_1, \xi_1) < 40c(d)\delta\sqrt{d} < D2^i\pi/(100\sqrt{d})$. By the fixed point Lemma 2.3.11 applied to the sphere $D2^i S^{d-1}$ we find a $w \in V_1 \cap 2^i D S^{d-1}$ fixed by the action of $h_1(\Gamma_\delta)$. Moreover, this vector is fixed by the action of $h(\Gamma_\delta)$. Also, $d_{S^{d-1}}(\xi_1, w) \le D2^i\pi/(100\sqrt{d})$. Thus

$$
\begin{aligned}
d_{S^{d-1}}(w, v) &\le d_{S^{d-1}}(v, \xi_1) + d_{S^{d-1}}(\xi_1, w) \qquad\qquad (2.4.3)\\
&\le D2^i\left(\pi/(100\sqrt{d}) + \pi/2 - \frac{1}{3\sqrt{d}}\right)\\
&\le D2^i\left(\pi/2 - \frac{1}{10\sqrt{d}}\right).
\end{aligned}
$$

The submanifold $\mathbb{R}^n \times \{w\}$ is invariant under the action of Γ_δ. Since the action of Γ_δ on $\mathbb{R}^n$ is also co-compact, the manifold $M_\delta = \mathbb{R}^n \times \{w\}/\Gamma_\delta$ is a compact flat manifold. Fix $R = D2^i(\pi/2 - \frac{1}{20\sqrt{d}})$. The normal exponential map gives a map of the R-ball B' in the tangent space of $D2^i S^{d-1}$ at w onto $B_w(R) \subset D2^i S^{d-1}$. Call this normal exponential map $e_w \colon \mathbb{R}^{d-1} \to 2^i D S^{d-1}$. When B' is equipped with the induced Euclidean metric, this map becomes a bi-Lipschitz map $B' \to B_w(R)$. (See the calculation for lens spaces in section 2.2, which can be generalized to this setting using polar co-ordinates). Taking products with $\mathbb{R}^n$ we get a bi-Lipschitz map $F \colon \mathbb{R}^n \times B' \to N_R(\mathbb{R}^n \times \{w\})$, where $N_R(\mathbb{R}^n \times \{w\})$ is the R-tubular neighborhood of $\mathbb{R}^n \times \{w\}$ in $\mathbb{R}^n \times 2^i D S^{d-1}$.

The action of Γ_δ induces an action on the normal bundle of the subset $\mathbb{R}^n \times \{w\}$, and thus on $\mathbb{R}^n \times B'$. Further, this action of Γ_δ on $\mathbb{R}^n \times B' \subset \mathbb{R}^n \times \mathbb{R}^{d-1}$ commutes

with the action on $\mathbb{R}^n \times 2^i DS^{d-1}$ via the map F. Thus, F induced a bi-Lipschitz map $f\colon \mathbb{R}^n \times B'/\Gamma_\delta \to N_R(\mathbb{R}^n \times \{w\})$. Use (2.4.3) to get that $B_{[(q,v)]_\delta}(\delta)$ is contained in $N_R(\mathbb{R}^n \times \{w\})/\Gamma_\delta$. Thus, the restricted inverse $f^{-1}\colon B_{[p,v]_\delta}(\delta) \to \mathbb{R}^n \times B'/\Gamma_\delta$ is a bi-Lipschitz map. The image $\mathbb{R}^n \times B'/\Gamma_\delta$ is an isometric subset of $\mathbb{R}^n \times \mathbb{R}^{d-1}/\Gamma_\delta$ which is a $d-1$-dimensional locally flat vector bundle over M_δ, which is a compact flat manifold of dimension n. Thus, it can be embedded using the statement $V - (n, d-1)$, which has smaller total dimension.

The other case is when V_1 is a $2k$-dimensional rotational subspace where the action of L is given by the Hopf-action. We can define Λ_L and $\Lambda_0 = \Lambda_L \cap \Gamma_\delta$ similarly to before. This time the action of Λ_0 on $\mathbb{R}^n$ will not necessary be trivial, but will be a Hopf-type action. For any point $\xi \in 2^i DS^{2k-1} \cap V_1$ its Λ_0-orbit is contained in a unique Hopf-fiber $S_\xi^1 \subset 2^i DS^{2k-1}$. The action of Γ_δ conjugates the Hopf-action to itself, and thus any orbit S_ξ^1 is mapped to another orbit S_ψ^1. We realize this orbit space as the complex projective space of diameter $2^i D$, i.e. $2^i D\mathbb{CP}^{(k-2)/2}$, and the action of Γ_δ descends to $2^i D\mathbb{CP}^{(k-2)/2}$. Denote by $S_{\xi_1}^1$ the orbit through ξ_1.

Apply the argument in Lemma 2.3.13 with $\Lambda = \Lambda_0$, $\Gamma = \Gamma_\delta$, $l = 24\delta\sqrt{d}$, and the seminorm $|g|' = d(gS_{\xi_1}^1, S_{\xi_1}^1)$. This gives that $\Lambda_0 \cap \Gamma_\delta$ has finite index in Γ_δ and any coset of Λ_0 in Γ_δ can be represented by an element $g \in \Gamma_\delta$ such that

$$d(gS_{\xi_1}^1, S_{\xi_1}^1) \leq d(g\xi_1, \xi_1) \leq 24\delta\sqrt{d}c(d) \tag{2.4.4}$$

Thus, the action of Γ_δ on $2^i D\mathbb{CP}^{(k-2)/2}$ is by a finite group $H_0 = \Gamma_0/\Lambda_0$. By (2.4.4), we have for any $h \in H_0$ that for the orbit $S_{\xi_1}^1$ corresponding to ξ_1:

$$d(hS_{\xi_1}^1, S_{\xi_1}^1) \leq d(g\xi_1, \xi_1) \leq 24c(d)\delta\sqrt{d} < D2^i\pi/4.$$

Lemma 2.3.11 can be used to give an orbit S_w^1 for some $w \in 2^i DS^{d-1}$, which is fixed under the action of H_0. In particular, the orbit as a subset of S^{d-1} is invariant under Γ_δ. Again, by Lemma 2.3.11 and (2.4.4) we get

$$d_{S^{d-1}}(S_w^1, v) \leq D2^i(\pi/2 - \frac{1}{10\sqrt{d}}). \tag{2.4.5}$$

This time define $M_\delta = \mathbb{R}^n \times S_w^1/\Gamma_\delta$ and consider it as a subset of K. Define $R = D2^i(\pi/2 - \frac{1}{20\sqrt{d}})$. The ball $B_{(q,v)}(\delta) \subset K$ is contained in the R-tubular neighborhood $N_R(\mathbb{R}^n \times S_w^1/\Gamma_\delta)$ by (2.4.5). See Figure 2.1 for a simplified image of this normal neighborhood. The R-tubular neighborhood can be identified in a bi-Lipschitz fashion with $\mathbb{R}^n \times S_w^1 \times B'/\Gamma_\delta$ for B' a Euclidean ball of radius R and dimension $d - 2$. The identification is via the normal exponential map of the submanifold $\mathbb{R}^n \times S_w^1/\Gamma_\delta \subset K$. The quotient $\mathbb{R}^n \times S_w^1 \times B'/\Gamma_\delta$ is isometric to a subset of a $d - 2$-dimensional locally flat vector bundle V'' over M_δ. In particular, we get a map $F \colon B_{(q,v)}(\delta) \to V''$. The conclusion thus follows from the statement $V - (n+1, d-2)$.

The previous paragraphs give bi-Lipschitz maps for each T_i when $i \geq 1$. For $i = 0$ we need a different argument. First assume $S = 1$. Similar to the case $V - (n, 0)$ when $S = 1$, we can use Lemma 2.3.5 and embed each ball $B_{[p]}(\delta)$ since for $\delta = l_1/(320c(n)L) \sim D$ we have $\Gamma_\delta = \{e\}$. In other words, the space has a lower bound on its injectivity radius. Also, note $\mathrm{Diam}\,(T_0) \leq 6D$.

Assume instead that $S > 1$. Consider an arbitrary $B_{[p]}(\delta)$ with $[p] \in T_0$ for $\delta = \max\left(\frac{l_S}{200Lc(n)}, 2l_{S-1}\right)$. Here $[p]$ is an orbit of the Γ action for $p = (p^*, v) \in \mathbb{R}^n \times \mathbb{R}^d$. Since $\mathrm{Diam}\,(T_0) \leq 6D \leq 200Lc(n)C(n)\delta$, we can use the Lemma 2.3.5 to construct embeddings once we can embed all of the balls $B_{[p]}(\delta)$. Again define
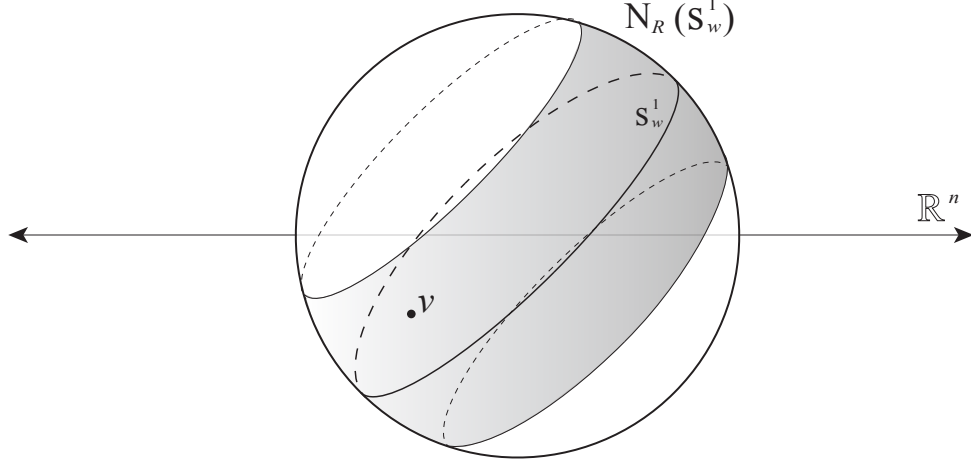
Figure 2.1: The normal neighborhood of an orbit of the Hopf action is presented in this simplified image, where S_w^1 is the invariant fiber, and the shaded region its normal neighborhood. We have simplified the setting by reducing the sphere to a 2-dimensional sphere, and by indicating the $\mathbb{R}^n$ factor abstractly as a line. The point v is depicted as belonging to the normal neighborhood. We thank Anthony Cardillo for drawing this figure.

$\Gamma_\delta = \langle g \in \Gamma | d(gp, p) \le 8\delta \rangle$. Also, define $\Gamma'_\delta = \langle g \in \Gamma | d(gp^*, p) \le 8\delta \rangle$. We have $\Gamma_\delta < \Gamma'_\delta$. Denote their orbits containing x by $[x]_\delta$ and $[x]'_\delta$ respectively. Then, we have $B_{[p]}(\delta) \subset V$ is isometric to $B_{[p]_\delta}(\delta) \subset \mathbb{R}^n \times \mathbb{R}^d/\Gamma_\delta$ and $B_{[p]'_\delta}(\delta) \subset \mathbb{R}^n \times \mathbb{R}^d/\Gamma'_\delta$. Here we used Lemma 2.3.8.

By the second case in the proof, there is an invariant affine subspace $O \subset \mathbb{R}^n$ of dimension $n - k$ such that $\Gamma'_\delta O \subset O$. In particular $\mathbb{R}^n \times \mathbb{R}^d/\Gamma'_\delta$ is isometric to $O \times \mathbb{R}^{d+k}/\Gamma'_\delta$. The space $O \times \mathbb{R}^{d+k}/\Gamma'_\delta$ is a locally flat vector bundle over O/Γ_δ,

and since $B_{[p]}(\delta)$ is isometric to a subset of it, we can embed it by the statement $V - (n - k, d + k)$ for some $k > 0$.

$$\square$$

2.4.2 Flat orbifolds

We next modify the proof slightly to allow us to embed complete flat orbifolds.

Theorem 2.4.6. *Every complete connected flat orbifold O of dimension n admits a bi-Lipschitz embedding into Euclidean space with distortion and dimension depending only on the dimension n. Further, every locally flat orbivector bundle over such a base with its natural metric admits such an embedding with distortion and target dimension depending only on the total dimension.*

Proof: We first give a description of the induction argument, and then move onto defining the relevant notation and proving the induction claims.

Summary of induction argument: The arguments are sufficiently similar to the proof of Theorem 2.4.1 that we only indicate the main differences. We will need to modify the induction statements slightly to add the case $V - (0, d)$, which corresponds to an orbifold over a point, i.e. $\mathbb{R}^n / \Gamma$, where Γ is a finite group. We assume the same notational conventions as used in the beginning of Theorem 2.4.1.

- $V - (n, 0)$: There exist constants $L(n, 0)$ and $N(n, 0)$ such that the following holds. Assume M is an arbitrary connected compact flat orbifold with dimension n. Then, there exists a bi-Lipschitz embedding $f \colon M \to \mathbb{R}^N$ with

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distortion $L \leq L(n, 0)$ and target dimension $N \leq N(n, 0)$ depending only on n.

- $V - (n, d), n \geq 1, d > 1$: There exist constants $L(n, d)$ and $N(n, d)$ such that the following holds. Assume V is an arbitrary d-dimensional locally flat Riemannian orbivector bundle over a compact flat orbifold M of dimension n. Then, there exists a bi-Lipschitz embedding $f \colon V \to \mathbb{R}^N$ with distortion $L \leq L(n, d)$ and target dimension $M \leq N(n, d)$ depending only on n, d.

- $V - (0, d)$: There exist constants $L(0, d)$ and $N(0, d)$ such that the following holds. Assume $V = \mathbb{R}^d / \Gamma$ where $\Gamma < O(d)$ is a finite group. Then, there exists a bi-Lipschitz embedding $f \colon V \to \mathbb{R}^N$ with distortion $L \leq L(0, d)$ and target dimension $N \leq N(0, d)$ depending only on d.

Overview of induction: We need to embed all complete connected flat orbifolds. Again, by Biberbach 's theorem every complete connected flat orbifold can be represented as a quotient by a discrete group of isometries 2.3.14. In particular, any such manifold is either isometric to $\mathbb{R}^n$, a quotient of $\mathbb{R}^n$ by a co-compact quotient or a locally flat vector bundle over such a base. In the first case the embedding is given by the identity mapping. Otherwise, either the space can be represented as $M = \mathbb{R}^n / \Gamma$, or $V = \mathbb{R}^n \times \mathbb{R}^d / \Gamma$, where Γ acts co-compactly, properly discontinuously and by isometries on $\mathbb{R}^n$ and by isometries on $\mathbb{R}^d$. The case $V - (n, 0)$ covers the problem of embedding M, and $V - (n, d)$ covers the problem of embedding V. We allow for $n = 0$ in these statement.

The induction workd in the same way, except we have new base cases for the induction $V - (0, 1)$ and $V - (0, 2)$.

Remark on notation: We will assume the same notation as in Theorem 2.4.1, except for the slight variation that the group actions are not assumed to be free.

Special cases $V - (0,1)$**,** $V - (0,2)$ **and** $V - (1,0)$: The spaces that result can all be enumerated and checked individually: a ray, two dimensional cone over a circle, a circle, or a cone over an interval. Their bi-Lipschitz embeddings can be explicitly constructed.

The rest of the proof covers various inductive steps similar to before, but including the degenerate one where the base is zero-dimensional.

Reduction $V - (0,n)$ **to** $V - (n-k, k-1)$**:** Let $\Gamma < O(n)$ be a finite subgroup. The space $M = \mathbb{R}^n/\Gamma$ is isometric to $C(S^{n-1}/\Gamma)$, i.e. the cone over S^{n-1}/Γ. By Lemma 2.3.4, it is sufficient to embed $S = S^{n-1}/\Gamma$. The proof of this claim is similar to the vector bundle reductions considered in the previous section. By the Jordan theorem 2.3.28 we have a finite index normal subgroup $K \triangleleft \Gamma$ which is abelian with index bounded by $[\Gamma : K] \leq c(n)^7$. Decompose $\mathbb{R}^n$ to canonical invariant subspaces V_i for K.

Let $\delta = \frac{1}{10^5 c(n)\sqrt{n}}$, and fix an arbitrary $[v] \in S^{n-1}/\Gamma$ which is represented by an element $v \in S^{n-1}$. By Lemma 2.3.5 it will be sufficient to embed $B_{[v]}(\delta)$. Define $\Gamma_\delta = \langle g \in \Gamma | d(gv, v) \leq 8\delta \rangle$, $K_\delta = \langle g \in K | d(gv, v) \leq 8\delta \rangle$ and $K'_\delta = \Gamma_\delta \cap K$. Decompose v into its components $v_i \in V_i$. There is an index i such that $||v_i|| \geq \frac{1}{3\sqrt{n}}$. To reduce clutter, reindex so that $i = 1$, and define $\xi_1 = \frac{v_1}{||v_1||}$. But then $d(g\xi_1, \xi_1) \leq 24\sqrt{n}\delta$. Repeating the argument in the proof of Theorem 2.4.1 we get either an element $w \in S^{n-1}$ fixed by Γ_δ, or a Hopf-orbit $S^1_w \subset V_1 \cap S^{n-1}$ through w which is invariant under Γ_δ. By using the normal exponential map, we get that

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[7]In this case Λ is the trivial group, and the statement is the classical Jordan theorem [138]

$B_{[v]}(\delta) \subset S^{n-1}$ is bi-Lipschitz to a subset of either a $n-1$ dimensional locally flat orbivector bundle over a point (i.e. w), or a $n-2$ dimensional orbivector bundle over a one dimensional compact flat orbifold (i.e. S^1_w/Γ_δ). These can be embedded using the statements $V-(0, n-1)$ or $V-(1, n-2)$.

Reduction $V-(n,0)$ to $V-(0,n)$ when $S=1$: This is very similar to the $V-(n,0)$ case in Theorem 2.4.1. We still have the estimate Diam $(M) < C(n)|l_1|$. Let $p \in \mathbb{R}^n$ be arbitrary. Set $\delta = l_1/(320c(n)L)$. Define the group $\Gamma_\delta = \langle g \in \Gamma | d(gp, p) < 8\delta \rangle$. On the other hand, by Lemma 2.3.13 for any $[p] \in M$ we have $B_{[p]}(\delta) \subset M$ is isometric to $B_{[p]_\delta}(\delta) \subset \mathbb{R}^n/\Gamma_\delta(p)$. Thus, by Lemma 2.3.5 we can reduce the embedding problem to finding embeddings for $B_{[p]_\delta}(\delta)$. Define $\Lambda_0 = \langle \lambda \in \Lambda | d(\lambda p, p) < 8\delta \rangle = \{e\}$, and similarly to before we get

$$0 \to 0 \to \Gamma_\delta \to H_0 \to 0,$$

where H_0 and thus Γ_δ is finite. The group Γ_δ has a fixed point o by Lemma 2.3.11, and thus we can identify $\mathbb{R}^n/\Gamma_\delta$ by a quotient of isometries fixing o. Thus, $\mathbb{R}^n/\Gamma_\delta$ can be embedded by reduction to the $V-(0,n)$-case above.

Reduction $V-(n,0)$ to $V-(n-k,k)$ when $S > 1$: The proof for manifolds translates almost verbatim. Consider $\delta = \max\left(\frac{l_S}{200Lc(n)}, 2l_{S-1}\right)$. Again Diam $(M) < C(n)l_S \leq 200Lc(n)C(n)\delta$ by Lemma 2.5.4. Thus, it is sufficient by Lemma 2.3.5 to embed each ball $B_{[p]}(\delta)$. Next fix $[p] \in M$, and its representative $p \in \mathbb{R}^n$. Repeating the arguments from the proof of Theorem 2.4.1 we can show that $B_{[p]}(\delta)$ is isometric to an orbivector bundle over a $(n-k)$-dimensional flat orbifold O. The only difference is that the action is not free.

Reduction $V - (n, d)$ to $V - (n-k, k+d), V - (a, b)$ with $k > 0$ or $a+b < n+d$.
There is no major change to the manifold case as we didn't use the fact that the action is free in any substantive way. The same decomposition into T_i's is used, followed by a consideration of the action of Γ_δ on S^{d-1}, and finding a fixed point w or a fixed Hopf-orbit S_ξ^1.

$\square$

Remark on bounds: The methods are likely not optimal, so we only give a rough estimate of the size of the dimension and distortion in the previous construction. Similar bounds could be obtained for the $\epsilon(n)$-quasiflat spaces considered below. Let $D(n)$ be the worst case distortion for a flat orbifold of dimension n, and let $N(n)$ be the worst case dimension needed. In particular $D(n) \le \max_{k=0,\dots,n} L(n - k, k)$, and $N(n) \le \max_{k=0,\dots,n} N(n - k, k)$, where L, N are defined in the statement of $V - (n, k)$ after Theorem 2.4.6. For dimension $n = 1$, we can bound the distortion by constants $L(1, 0)$, and $L(0, 1)$, and they can be embedded in 3 dimensions.

Next, $V - (n, d)$ was reduced to $V - (n-k, d+k)$ and $V - (a, b)$ with $a+b < n+d$. Each such reduction is done by using Lemma 2.3.5 at the scale $R/r \sim n!^{n+1} \sim O\left(e^{Cn^2 \ln(n)}\right)$, which increases distortion by a multiplicative factor of $O\left(e^{Cn^2 \ln(n)}\right)$. In order to reduce $V - (n, d)$ to lower dimensional spaces, this reduction might need to be repeated up to n times. Thus, $D(n) = O\left(e^{Cn^3 \ln(n)} D(n - 1)\right)$, which gives $D(n) = O\left(e^{Cn^4 \ln(n)}\right)$. By a similar analysis, we get that the dimension of the embedding can be bounded by $N(n) = O\left(e^{Cn \ln(n)}\right)$. Our arguments are not optimal and probably some exponential factor could be removed, but the methods don't seem to yield better than super exponential distortion.

2.4.3 Quasiflat orbifolds

Below, the proof of Theorem 2.1.2 will proceed by approximating the orbifold by a flat orbifold. This will involve quotienting away the commutator subgroup $[N, N]$ and using Lemma 2.3.6. Thus, we begin by studying lattices and their commutators.

Lemma 2.4.7. *Let $\Lambda < N$ be a lattice. Then $[\Lambda, \Lambda]$ is a lattice in $[N, N]$.*

Proof: $\Lambda' = [\Lambda, \Lambda]$ is a torsion free nilpotent Lie group and by Malcev it corresponds to a connected Lie group N' such that N'/Λ' is compact (see [102] and [31]). Further by Malcev, there is a unique homomorphism $f \colon N' \to N$ such that $f_*(\Lambda') = \Lambda'$. We want to show that $f(N') = [N, N]$. First of all, since $\Lambda' < [N, N]$, we get that $f(N') < [N, N]$.

The image $f(N')$ is normal in N, and $N/f(N')$ is a nilpotent Lie group. The normality follows since Λ' is normal in Λ, and thus $\lambda f(N') \lambda^{-1} < N'$ for all $\lambda \in \Lambda$. From this we can conclude, using the nilpotency of N' and the fact that Λ is a lattice, that $N f(N') N^{-1} < N'$. This follows since every element can be represented as $n = \prod_i e^{t_i X_i}$. The derivative of $\mathrm{Ad}_n(x) = nxn^{-1}$ at the identity is $\mathrm{ad}_n(t_i)$, which is a matrix-polynomial in t_i, which has the tangent space of $f(N')$ as an invariant subspace whenever t_i is integer. Thus, the same tangent space is left invariant by $\mathrm{ad}_n(t_i)$ for any $t_i \in \mathbb{R}$.

The projection $\pi \colon N/\Lambda' \to N/f(N')$ has compact fibers and thus we can show that the action of $\Lambda/[\Lambda, \Lambda]$ on $N/f(N')$ is properly discontinuous and co-compact. However, $\Lambda/[\Lambda, \Lambda]$ is abelian, and thus $N/f(N')$ is also abelian (by another application of Malcev's theorems). Therefore, $[N, N] < f(N')$, which completes the proof.

Lemma 2.4.8. *Let n be an integer, and $\epsilon(n)$ be sufficiently small[8] and $l \leq \epsilon(n)$ fixed. There exists a dimension dependent constant $C(n)$ such that the following holds. If N is a n-dimensional simply connected nilpotent Lie group with a left invariant metric g and sectional curvature $|K| \leq 2$, and if $\Lambda < N$ is a lattice which is generated by elements of length $|l|_e \leq l \leq \epsilon(n)$, then Diam $(N/\Lambda) \leq C(n)l$.*

Proof: As in Lemma 2.3.23, we choose a short basis γ_i for the lattice Λ. By construction $|\gamma_i|_e \leq l$ for all $i = 1, \ldots, n$. The result then follows from Lemma 2.5.4, with the same choice of $C(n)$.

$\square$

Lemma 2.4.9. *Let S be an $\epsilon(n)$-quasiflat n-dimensional orbifold. Then there exist constants $0 < \delta < 1$ depending only on the dimension n and a constant $A(n)$, such that the following two properties hold.*

- *For every $[p] \in S$ there exists a flat Riemannian orbivector bundle $V = V(S')$ over a k-dimensional $\epsilon(k)$-quasiflat orbifold S' with $k < n$, with dimensions of the fibers $n - k$, and a $A(n)$-bi-Lipschitz map $f_{[p]} \colon B_{[p]}(\delta \, diam(S)) \to V(S')$.*

- *There exists a flat orbifold S'' such that $d_{GH}(S, S') < \delta \, diam(S)$.*

Proof: Represent S as a quotient $S = N/\Gamma$, where Γ is the discrete group guaranteed by Definition 2.3.17. Further, take the finite index subgroup $\Lambda = N \cap \Gamma$

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[8]See the remark at the beginning of Section 2.3.

with $[\Gamma : \Lambda] < c(n)$, which is guaranteed by Lemma 2.3.19. Let $l = 400c(n), L = 2l^n, I_j, J_k, \gamma_j, \Lambda_k, L_k$ and s be as in Lemma 2.3.23 and Definition 2.3.25. As in Lemma 2.3.23, we need to assume $\epsilon(n)$ small enough. For this proof, $\epsilon(n) \leq \frac{\delta(n)}{C(n)10^3 c(n)C'(n)L}$ is enough. See Section 2.3 for the definitions of the various constants. We will first prove the existence of a near-by flat orbifold.

As in Lemma 2.3.23, we can generate Λ by elements α_i such that $|\alpha_i|_e \leq 4c(n)$ Diam $(S) \leq 2c(n)\epsilon(n)$. As such, from nilpotency we see that $[\Lambda, \Lambda] = \{[\mu, \nu] | \mu, \nu \in \Lambda\} < \Lambda$ is generated by elements $\beta_{i,j} = [\alpha_i, \alpha_j] \in N$. From (2.3.24) we get $|\beta_{i,j}|_e <$ Diam $(S)/(10C(n)L^2) <$ Diam $(S)/(8LC(n)c(n))$.

Choose the largest index a such that $|\gamma_a|_e <$ Diam $(S)/(8LC(n)c(n))$, and let $a \in I_b$ for some b. By construction $l_b \leq L|\gamma_a|_e \leq$ Diam $(S)/(8C(n)c(n))$. Also, $l_b < l_s$ because $\text{diam}(S) \leq C(n)l_s$ by Lemma 2.5.4. Thus, $L_b \neq N$.

We have that $\beta_{i,j} \in \Lambda_b$, because if $\beta_{i,j} \notin \Lambda_b$, then, by construction, $i_b + 1 \notin J_b$ and we have (by construction of the basis in Lemma 2.3.23)

$$|\gamma_{i_b+1}|_e \leq |\beta_{i,j}|_e < \text{diam}(S)/(8Lc(n)C(n)),$$

which would contradict the choice of a. Thus, we see that $[\Lambda, \Lambda] < L_b$ and that $l_b \leq \text{diam}(S)/(8C(n)c(n))$. Since L_b is invariant under conjugation by elements in Γ and the holonomy of Γ, we see that the action of Γ descends to an action of a discrete group Γ' on $N/L_b = N'$, where $\Gamma' = \Gamma/\Gamma_b$ and Γ_b is the isotropy group of L_b. Define $N'/\Gamma' = S'$. Note that Γ' has a translational subgroup $\Lambda' = \Lambda/\Lambda_b$ which is abelian and acts co-compactly on N'. Therefore, N' is in fact flat and S' is a flat orbifold. We will next show that $d_{GH}(S', S) < \delta\text{diam}(S)$ for $\delta = 1/(2c(n))$.

The orbits of the L_b-action on N descend to orbits on S, and S' can be identified by the orbit-space of the L_b-orbits on S. This gives a map between orbifolds

$\pi\colon S \to S'$, which lifts to a Riemannian submersion on their orbifold universal covers, which are N and N' respectively. We have Diam $(S') \le$ Diam $(S) \le \epsilon(n)$. Choose a coset $[p]' \in S'$ represented by an element $p \in N$ with

$$|p| \le 4c(n)\epsilon(n). \tag{2.4.10}$$

Denote by $[p] = \pi^{-1}([p]')$ the coset in S.

Thus, it will be sufficient to prove that Diam $(\pi^{-1}([p])) < \delta \mathrm{diam}(S)/2$. We see that $\pi^{-1}([p]') = O_{[p]}$, where $O_{[p]} \subset S$ is the orbit of L_b passing through $[p] \in S$. The set $O_{[p]}$ is isometric to L_b/Γ_p where $\Gamma_p = \{p^{-1}\gamma p | \gamma \in \Gamma\}$ with a translational subgroup $\Lambda_p = p^{-1}\Lambda'_b p$. This translational subgroup has a basis given by $p^{-1}\gamma_i p$ for $i = 1, \ldots, i_{b+1} - 1$. We have, by (2.3.24) and (2.4.10) that

$$|p^{-1}\gamma_i p|_e \le 2|\gamma_i|_e \le \mathrm{diam}(S)/(4C(n)c(n)).$$

By Lemma 2.4.8 we have Diam $(L_b/\Gamma_p) \le \mathrm{diam}(S)/(4c(n))$. Thus, $d_{GH}(S', S) \le \delta \mathrm{diam}(S)$ follows by our choice $\delta = 1/(2c(n))$ and using Lemma 2.3.7.

Next, we prove the first part, that at the scale of $\delta' = \delta \mathrm{diam}(S)$ the space has a description in terms of a simpler vector bundle. Define $\Gamma_{\delta'}(p) = \langle \gamma \in \Gamma | d(\gamma p, p) \le 8\delta' \rangle$. By Lemma 2.3.8 we know that $B_{[p]}(\delta') \subset S$ is isometric to $B_{[p]_{\delta'}}(\delta') \subset N/\Gamma_{\delta'}(p)$. Further by Lemma 2.3.13, and observing that $\Lambda_{\delta'} = \langle \gamma \in \Lambda | d(\gamma p, p) \le 8\delta' \rangle = \Lambda_b$ we have a finite group H_0 and a short exact sequence

$$0 \to \Lambda_b \to \Gamma_{\delta'} \to H_0 \to 0.$$

The action of H_0 descends to an action on N/L_b by $h[x]_{L_b} = [hx]_{L_b}$. Further,

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by Lemma 2.3.13, for every element $h \in H_0$ we can choose a representative $\gamma_h \in \Gamma_{\delta'}$ with $d(\gamma_h p, p) \leq 8(c(n) + 1)\delta'$. In particular $d(h[p]_{L_b}, [p]_{L_b}) \leq 8(c(n) + 1)\delta'$. Thus, by Lemma 2.3.11 used on N/L_b (which is flat) there is an orbit $[q]_{L_b}$ which is fixed by the action of H_0 and $d([q]_{L_b}, p) \leq 8(c(n) + 1)\delta'$. Further, the orbit $[q]_{L_b} \subset N$ is invariant under the action of $\Gamma_{\delta'}$.

Consider the normal bundle $\nu([q]_{L_b})$ of $[q]_{L_b}$ in N. The bundle has a trivialization and can be identified with $\mathbb{R}^t \times L_b$ for some t. This trivialization can be obtained similarly to Lemma 2.5.5. By Lemma 2.5.5 there is a map $f \colon N_{2\delta(n)}([q]_{L_b}) \to \mathbb{R}^t \times L_b$ from the $2\delta(n)$-tubular neighborhood $N_{2\delta(n)}([q]_{L_b})$ of $[q]_{L_b}$. This map is also $A(n)$-bi-Lipschitz on the ball $B_q(2\delta(n))$. Further, the group $\Gamma_{\delta'}$ acts on $N_{2\delta(n)}([q]_{L_b})$ and $\mathbb{R}^t \times L_b$ by isometries, and the action commutes with f. We can thus show that restricted to the ball $B = B_{[q]_{\delta'}}(\delta(n)) \subset N/\Gamma_{\delta'}$ the induced map $\overline{f} \colon N_{2\delta(n)}([q]_{L_b})/\Gamma_{\delta'} \to \mathbb{R}^t \times L_b/\Gamma_{\delta'}$ is also $A(n)$-bi-Lipschitz.

Denote the orbifold by $L_b/\Gamma_\delta = S'$, and its dimension by $k < n$. The orbifold S' is lower dimensional because $L_b < N$ strictly. The space $\mathbb{R}^t \times L_b/\Gamma_\delta = V(S')$ is a locally flat Riemannian orbivector bundle over the k-dimensional $\epsilon(k)-$quasiflat orbifold S'[9]. Also, $B_{[p]_{\delta'}}(\delta') \subset B_{[q]_{\delta'}}(\delta(n))$, so further restricting we get a $A(n)$-bi-Lipschitz map $\overline{f} \colon B_{[p]_{\delta'}}(\delta') \to V(S')$, which is our desired map.

$\square$

A similar statement is needed for orbivector bundles.

Lemma 2.4.11. *Let $\epsilon(n)$ be small enough. For every d and n there are constants $A(n, d)$ such that the following holds. Let $V = V(S)$ be a locally flat Riemannian*

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[9]S' will have the same bound on its curvature and diameter as S. We use here the assumption that $\epsilon(k) \geq \epsilon(n)$ for $k \leq n$.

d-dimensional orbivector bundle over an $\epsilon(n)$-quasiflat n-dimensional orbifold S.
Then there exist constants $0 < \delta, \delta' < 1$, which depend only on the dimension n
and d, such that the following two properties hold.

- *For every $p \in V(S)$ there exists a flat Riemannian orbivector $V = V(S')$*
  bundle over a k-dimensional $\epsilon(k)$-quasiflat orbifold S' with $k < n$ and a
  $A(n, d)$ bi-Lipschitz map $f_p \colon B_p(\delta \operatorname{Diam}(S)) \to V(S')$.

- *There exists a locally flat Riemannian orbivector bundle $V'' = V(S'')$ over a*
  flat basis S'' such that $d_{GH}(V, V'') < \delta' diam(S)$.

Proof: For this proof $\epsilon(n) \leq \frac{\delta(n)}{C(n)10^3 c(n)C'(n)L}$ is enough (see Definition 2.3.25 for the definition of L). We can represent the spaces as $S = N/\Gamma$ and $V = N \times \mathbb{R}^d/\Gamma$, where Γ acts by rotations on $\mathbb{R}^d$ and co-compactly on N so that S is $\epsilon(n)$-quasiflat. By Lemma 2.3.19, there is a lattice $\Lambda = \Gamma \cap N$ of N with $[\Gamma : \Lambda] \leq c(n)$. Further, there is an induced action $h \colon \Lambda \to O(d)$. By Lemma 2.3.28 we have an index $c(d)$ subgroup $\Lambda' < \Lambda$ with $[\Lambda : \Lambda'] \leq c(d)$ and $h(\Lambda')$ abelian. In particular $[\Lambda', \Lambda'] = \Lambda'_c < \Lambda'$.

The group Λ'_c acts trivially on $\mathbb{R}^d$ and by translations on N. Consider the left action of N_c on $N \times \mathbb{R}^d$, which acts trivially on $\mathbb{R}^d$. The group Γ maps orbits of N_c to similar orbits, and thus $\Gamma_c = \Gamma/\Lambda'_c$ is a discrete group which acts properly discontinuously on $N/N_c \times \mathbb{R}^d$. Further, the orbits of N_c descend to V. Denote by $V_c = (N/N_c \times \mathbb{R}^d)/\Gamma_c$ the group quotient, and $\pi \colon V \to V_c$ the induced projection map. The group action of Γ_c is co-compact on N/N_c, and by rotations on $\mathbb{R}^d$. Further, N/N_c is abelian, and thus isomorphic and isometric to $\mathbb{R}^l$ for some l. In other words, up to an isometry $V_c = \mathbb{R}^l \times \mathbb{R}^d/\Gamma_c$, which is a d-dimensional locally flat orbivector bundle over the flat orbifold $\mathbb{R}^l/\Gamma_c$.

Next, we give a diameter bound needed to apply Lemma 2.3.7. Take any $q = [(p, v)] \in V$ and the N_c orbit O_q passing through that point. We can assume, as in Lemma 2.4.9, that $|p|_e \leq 2c(n)\epsilon(n)$. Since Λ' is a lattice of N by Lemma 2.4.7, also Λ'_c is a lattice in $[N, N] = N_c$. Also $O_q = N_c/p^{-1}Lambda_c p$. The compact manifold O_q is a subset of $N/p^{-1}\Lambda'p$, and $N_c/p^{-1}[\Lambda, \Lambda]p$ is a compact manifold with Diam $(N_c/[\Lambda, \Lambda]) \leq 20C(n)$ Diam $(S)c(n)$ by computations from Lemma 2.4.9. Further, $N_c/p^{-1}[\Lambda, \Lambda]p$ covers O_q with index at most $c(d)$. Thus,

$$\text{Diam } (O_q) \leq 20C(n) \text{ Diam } (S)c(n)c(d). \qquad (2.4.12)$$

Note that O_q can be covered by $N \times \{(p, v)\}/\Lambda_c$, where (p, v) is a representative of q in $N \times \mathbb{R}^d$. Also, the action of Λ_c is trivial on $\mathbb{R}^d$. Define $\delta' = 40C(n)c(n)c(d)$ and $\Delta = 40C(n)c(n)c(d)$ Diam (S). As before, we can show that π is a Δ-Gromov-Hausdorff approximation. In particular $d_{GH}(V_c, V) \leq \Delta$. This shows the second part.

We will next show the first statement by a similar argument as in Lemma 2.4.9. Fix an arbitrary $p = (q, v) \in N \times \mathbb{R}^d$. Let $l = 400c(n), L = 2l^n, I_j, J_k, \gamma_j, \Lambda_k, L_k$ and s be as in Lemma 2.3.23 and Definition 2.3.25. As in Lemma 2.3.23, we need to assume $\epsilon(n)$ small enough.

As in Lemma 2.3.23, choose the largest index a such that $|\gamma_a|_e <$ Diam $(S)/(8Lc(n)C(n))$, and let $a \in I_b$ for some b. By construction $l_b \leq L|\gamma_a|_e \leq$ Diam $(S)/(8c(n)C(n))$ and $l_b < l_s$.

Let $\delta = 1/(8c(n))$ and redefine $\Delta = \delta$ Diam (S). Define $\Gamma_\Delta(q) = \{\gamma \in \Gamma | d(\gamma q, q) \leq 8\Delta\}$ and $\Gamma_\Delta(p) = \{\gamma \in \Gamma | d(\gamma p, p) \leq 8\Delta\}$. Since $\Gamma_\Delta(p) < \Gamma_\Delta(q)$, by Lemma 2.3.8 we know that $B_{[p]}(\Delta) \subset V$ is isometric to $B_{[p]_\Delta}(\Delta) \subset N \times \mathbb{R}^d/\Gamma_\Delta(q)$. Note, we first show that it is isometric to a ball in $N/\Gamma_\Delta(p)$, but by the inclusion

$\Gamma_\Delta(p) < \Gamma_\Delta(q)$ it is easy to see that this induces the desired isometry. Further, by Lemma 2.3.13 we have a finite group H_0 and a short exact sequence

$$0 \to \Lambda_b \to \Gamma_\Delta(q) \to H_0 \to 0.$$

Here, $\Lambda_b < N$ is defined similarly to Lemma 2.4.9. The group Λ_b is a lattice in the connected Lie subgroup $L_b < N$ associated to it (see Lemma 2.3.22). By the same argument as in Lemma 2.4.9, we can find a L_b-orbit $[x]_{L_b}$ passing through $x \in N$ with $d(x, q) < 8(c(n) + 1)\delta'$ in N which is invariant under the action of Γ. By Lemma 2.5.5, we find for the $\delta(n)$-tubular neighborhood $N_{\delta(n)}([x]_{L_b})$ of $[x]_{L_b}$ in N a map $f \colon N_{\delta(n)}([x]_{L_b}) \times \mathbb{R}^d \to L_b \times \mathbb{R}^t \times \mathbb{R}^d$, which is identity on the $\mathbb{R}^d$ factor. Further, f is $A(n)$-bi-Lipschitz when restricted to $B_x(2\delta(n)) \times \mathbb{R}^d$.

The action of the group $\Gamma_\Delta(q)$ induces canonically an action on $L_b \times \mathbb{R}^t \times \mathbb{R}^d$. Quotienting by this action induces a map $\overline{f} \colon N_{\delta(n)}(([x]_{L_b})) \times \mathbb{R}^d / \Gamma_\Delta(p) \to L_b \times \mathbb{R}^t \times \mathbb{R}^d / \Gamma_\Delta(p)$. Let $B(\delta(n)) = N_{\delta(n)}(([x]_{L_b})) \times \mathbb{R}^d / \Gamma_\Delta(p)$. The space $V'' = L_b \times \mathbb{R}^t \times \mathbb{R}^d / \Gamma_\Delta(p)$ is a locally flat Riemannian orbivector bundle over $L_b / \Gamma_\Delta(p)$, which is an $\epsilon(n-l)$-quasiflat $n-l$-dimensional orbifold for some $n \geq l > 0$. Note that $\epsilon(n-l) \geq \epsilon(n)$ by assumption.

Finally, consider the ball $B_{[p]_\Delta}(3\Delta) \subset B(\delta(n))$. This inclusion can be seen by considering lifted balls and noting that $B_p(2\Delta) \subset A(\delta(n))$. Note that we have chosen $\epsilon(n)$ so small that $8(c(n)+2)\Delta < \frac{\delta(n)}{10}$. The restricted map $\overline{f}|_{B_{[p]_\Delta}(\Delta)}$ is also $A(n)$-bi-Lipschitz.

$\square$

Finally, we can prove the embedding theorem for quasiflat manifolds and vector

bundles.

Proof of Theorem 2.1.2: The proof will proceed by a reverse induction argument initiated by the flat case of the theorem. We will define the induction statements $E(d, n)$ as follows.

- $E(0, n)$: There exist $N(0, n)$ and $L(0, n)$ such that every $\epsilon(n)$-quasiflat n-dimensional orbifold S admits a bi-Lipschitz map $f \colon S \to \mathbb{R}^{N(0,n)}$ with distortion $L(0, n)$.

- $E(d, n)$ for $n \geq d \geq 1$: There exist $N(d, n)$ and $L(d, n)$ such that every d-dimensional locally flat orbivector bundle $V = V(S)$ over an $\epsilon(n-d)$-quasiflat $n-d$-dimensional orbifold S admits a bi-Lipschitz map $f \colon V \to \mathbb{R}^{N(d,n)}$ with distortion $L(d, n)$.

The statement will be proved by an induction on n. The case $n = 1$ is completed since the cases included in $E(0, 1)$ and $E(1, 1)$ are flat orbifolds and can be embedded using Theorem 2.4.6. Next, fix $n > 1$ and assume we have proved $E(t, s)$ for all $1 \leq s \leq n$ and $0 \leq t \leq s$.

We proceed to show $E(d, n)$ by reverse induction on d. The base cases are $d = n, n - 1$. The statement $E(n, n)$ corresponds to a flat orbifold and is covered in Theorem 2.4.6. Further, the case $E(n - 1, n)$ corresponds to a locally flat orbivector bundle over a flat base, so is also covered in Theorem 2.4.6.

Next, fix $0 \leq d \leq n - 2$. Assume the statement $E(k, n)$ has been proved for $k > d$. There are two slightly different cases $d = 0$ and $d \geq 1$. Assume the first. In this case by Lemma 2.4.9 there is a δ such that $B_p(\delta \operatorname{Diam}(S))$ is bi-Lipschitz to a case covered by $E(s, n)$ for some $s > 0$. Thus, $B_p(\delta \operatorname{Diam}(S))$ admits a bi-Lipschitz embedding by induction. Also, $d_{GH}(S, S') \leq \delta \operatorname{Diam}(S)$ where S' is a flat orbifold

of lower dimension. Also, S' admits a bi-Lischitz embedding by Theorems 2.4.6. The statement then follows from Lemma 2.3.6. The case $d \geq 1$ is similar, but one uses Lemma 2.4.11 instead and concludes that $B_p(\delta \text{ Diam } (S))$ is bi-Lipschitz to a case covered by $E(d+s,n)$ for some $s > 0$, and $d_{GH}(V,V') \leq \delta' \text{ Diam } (S)$ for a locally flat orbivector bundle V' over a flat base. The constants δ, δ' here depend only on the dimension.

We remark, that to apply Lemma 2.3.6 we technically need to show that every $\epsilon(n)$-quasiflat space is doubling. This could be shown combining techniques from [?, IV.5.8] (and [107]). Also, Lemmas 2.4.11 and 2.4.9 combined with induction actually can be used to prove Doubling directly. However, in order to apply the proof of Lemma 2.3.6 we only need the doubling estimate on all balls $B_p(R) \subset V$ and $B_p(R) \subset R$, where $R > 0$ has a fixed upper bound depending on $\epsilon(n), \delta, \delta'$ and the constants $N(s,t), L(s,t)$ for cases $E(s,t)$ assumed true by induction. This doubling estimate follows by the sectional curvature estimate $|K| \leq 2$.

$\square$

2.4.4   Proofs of main theorems

With the previous special cases at hand we can prove the main theorems.

Proof of Theorem 1.2.22: Fix $\epsilon = \epsilon(n)$ and $\eta = 1/4$. By Fukaya's fibration Theorem 2.1.3, every point $p \in M^n$ has a definite sized ball $B_p(\delta)$, where $\delta = 5\rho(n, \eta, \epsilon)$, equipped with a metric g' such that $(B_p(\delta), g')$ is isometric to a subset of a compact $\epsilon(n)$-quasiflat manifold M' or a locally flat vector bundle V over such a base. Since $||g - g'|| < 1/4$, then the ball $B_p(\delta/2)$ is bi-Lipschitz to a subset of V, or M', with distortion at most $\frac{1+1/4}{1-1/4}$. Since the spaces M' and V can be

embedded by Theorem 2.1.2, we can embed M^n using a doubling Lemma 2.3.5 with $R = \mathrm{Diam}\,(A)$ and $r = \delta$. Note, by our choice of ϵ and η, the constant δ depends only on the dimension n.

$\square$

Proof of Theorems 1.2.23 and 1.2.24: The only difference to the proof of Theorem 1.2.22 is using Fukaya's fibration theorem for orbifolds 2.1.4 with $\epsilon = \epsilon(n)$ and $\eta = 1/4$.

$\square$

2.5   Auxiliary results

2.5.1   Collapsing theory of orbifolds

Much of the terminology here is presented in [89, 139]. Recall, a *topological orbifold* is a second countable Hausdorff space with an atlas of charts U_i, which are open sets satisfying the following.

- Each U_i has a covering by $\hat{U}_i$ with continuous open embeddings $\phi_i \colon \hat{U}_i \to \mathbb{R}^n$ and a finite group G_i acting continuously on $\hat{U}_i$ such that U_i is homeomorphic to $\hat{U}_i/G_i$. Denote the projection by $\pi_i \colon \hat{U}_i \to U_i$.

- There are transition homeomorphisms $\phi_{ij} \colon \pi_i^{-1}(U_i \cap U_j) \to \pi_j^{-1}(U_i \cap U_j)$, such that $\pi_j \circ \phi_{ij} = \pi_i$.

A chart is denoted simply by U_i, without explicating the covering $\hat{U}_i$ or other aspects. An *orbifold* is defined as one for which $\phi_j \circ \phi_{ij} \circ \phi_i^{-1}$ are smooth maps. Further, a *Riemannian orbifold* is an orbifold with a metric tensor g_i on each U_i for which $d\phi_{ij}^*(g_j) = g_i$. Any other notion like geodesics, sectional curvature or Ricci-curvature, is defined by considering the lifted metric on $\hat{U}_i$. One may also define the notion of orbifold maps as those that lift to the covering charts $\hat{U}_i$ and are compatible with the projections. A smooth orbifold map is also naturally defined by considering charts. Such a map is non-singular if in charts it can be expressed as a local diffeomorphism. A non-singular orbifold map which is surjective is called an orbifold covering map.

We also need a notion of an orbivector bundle and an orbiframe bundle. An orbivector bundle is a triple (V, O, π), where V, O are smooth orbifolds and $\pi \colon V \to O$ is a smooth map with the following properties.

- There is an atlas of O consisting of charts U_i such that for all of the charts $(U_i, \hat{U}_i, G_i)$ of O there is a chart $V_i = \pi_i^{-1}(U_i)$

- There are covering charts $\hat{V}_i = \hat{U}_i \times \mathbb{R}^k$ and a G_i-action, such that $V_i = \hat{V}_i / G_i$, and maps $\pi_i^V \colon \hat{V}_i \to V_i$.

- The projections $\hat{\pi}_i \colon \hat{V}_i \to \hat{U}_i$ satisfy $\hat{\pi}_i = \pi \circ \pi_i^V$.

- The action of G_i on $\hat{U}_i \times \mathbb{R}^k$ splits as $g(u, x) = (gu, g_u x)$, where G_i acts linearly on $\mathbb{R}^k$ (the action can depend on the base point u), and the action on the first component is the original action.

- The actions of G_i and G_j on overlapping charts commute.

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If there is a bundle metric on each $\hat{V}_i$, such that G_i acts by isometries on the fibers $\mathbb{R}^k$ and such that it is invariant under transition maps, then we call the orbivector bundle a Riemannian orbivector bundle. A connection on such bundle is defined as an affine connection on each $\hat{V}_i$ which is invariant under G_i and the compatible via the transition maps. Such a connection is flat if it is flat on each chart. Such a connection together with a Riemannian structure on the base induces a metric on V making it into a Riemannian orbifold. This allows us to define a central notion in this chapter.

Definition 2.5.1. (V, O, π) is called a locally flat orbivector bundle, if it is a Riemannian orbivector bundle over a Riemannian orbifold with a flat connection.

To every smooth orbifold we can associate a tangent orbivector bundle TO, where the group actions on the fibers are given by differentials. The fiber of a point $\pi^{-1}(p) \subset TO$ is denoted $C_p(O)$. For Riemannian orbifolds, this is easily seen to correspond with the metric tangent cone at p. In that case $C_p(O) = \mathbb{R}^n/G$ for some group G acting by isometries on $\mathbb{R}^n$. One can also define orbifold principal bundles by considering a fiber modeled on a Lie group G and assuming transition maps are given by left-multiplication. This allows us to define for Riemannian orbifolds an orbiframe bundle FO. The orbiframe bundle FO is always a manifold.

In order to do collapsing theory, we need to define a local orbifold cover. A connected Riemannian orbifold has a distance function and we can define a complete Riemannian orbifold as one which is complete with respect to its distance function. On such orbifolds we may define an exponential map $\exp\colon TO \to O$ by considering geodesics and their lifts. Fix now a point $p \in O$ and take the exponential map at that point $\exp_p\colon C_p(O) \to O$, which is surjective. As $C_p(O) = \mathbb{R}^n/G$, we can lift the exponential map to a map $\widetilde{\exp}_p\colon \mathbb{R}^n \to O$. Next we state a crucial

property of such an exponential map.

Lemma 2.5.2. *If a complete Riemannian orbifold has sectional curvature bounded by $|K| \leq 1$, then $\widetilde{exp}_p$ is a non-singular orbifold map on the ball $B_0(\pi) \subset \mathbb{R}^n$.*

The map $\exp_p \colon C_p(O) \to O$ will define locally a homeomorphism in the neighborhood of $p \in C_p(O)$. We call the largest δ such that $\exp_p|_{B(p,\delta)}$ is injective on $B(p, \delta) \subset C_p(O)$ the *injectivity radius* at p, and denote it $\mathrm{inj}_p(O)$. If $O' \subset O$ is a sub-orbifold, then we can also define a normal bundle of it and a normal injectivity radius by the largest radius such that the normal exponential map is injective. These terms are used in the proof of Theorem 2.1.4 below.

Lemma 2.5.2 is alluded to in the appendix of [48] and can be proved by using local co-ordinates and standard Jacobi-field estimates. Using this terminology, we can now state and outline the proof of Fukaya's fibration theorem for collapsed orbifolds

Theorem 2.5.3. *Let (O, g) be a complete Riemannian orbifold of dimension n and sectional curvature $|K| \leq 1$, and $\epsilon > 0$ be an arbitrary constant. For every $\eta > 0$ there exists a universal $\rho(n, \eta, \epsilon) > 0$ such that for any point $p \in O$ there exists a metric g' on the ball $B_p(10\rho(n, \eta, \epsilon))$ and a complete Riemannian orbifold O' with the following properties.*

-

$$||g - g'||_g < \eta$$

- *$(B_p(5\rho(n, \eta, \epsilon)), g')$ is isometric to a subset of O'.*

- *O' is either a ϵ-quasiflat orbifold or a locally flat Riemannian orbivector bundle over a ϵ-quasiflat orbifold S.*

Proof: Fix $\epsilon, \eta > 0$. Without loss of generality, we can assume ϵ is very small. If the result were not to hold, then it would have to fail for a sequence of pointed orbifolds (O_i, p_i). Thus, let O_i be a sequence of examples such that there doesn't exist any $\delta > 0$ such that for any i the ball $B_{p_i}(\delta)$ has a metric g''_η with the following two properties. On the one hand $||g''_\eta - g||_g < \eta$, and $(B_{p_i}(\delta), g''_\eta)$ is isometric to a subset of an ϵ-quasiflat orbifold, or an orbivector bundle over such a base. We will see a contradiction by establishing the existence of such a δ.

One should first regularize the orbifold slightly as to attain uniform bounds on all co-variant derivatives. This is done in [89, Second part, Section 5]. The action of the local group on the frame bundle is free and thus Fukaya's proofs in [58, 59] (see also [31]) show that a subsequence of the orbifold frame bundles FO_i converges to a smooth manifold Y with bounds on all covariant derivatives. Assume without loss of generality that FO_i is already this sequence.

Using standard collapsing theory from Cheeger, Gromov and Fukaya [31], we obtain an equivariant fibration of FO_i by nilpotent fibers, and a metric g_η which is invariant under the local nilpotent action and the local $O(n)$ action. It is possible for those orbits to be finite. Further, we have bounds on their second fundamental forms and normal injectivity radii of the nilpotent fibers. (In the case of a point, the normal injectivity radius is just the injectivity radius.) By a remark in [123], we can also control the sectional curvatures of g_η. By equivariance, this structure descends onto a fibration of O_i by orbits of nilpotent Lie groups. (Again, the fibers may be points, in which case the normal injectivity radius bound means the injectivity radius bound). Corresponding to this fibration we have, after possibly passing to another subsequence and re-indexing, an invariant metric g'_η, which satisfies $||g - g'_\eta||_g < \eta'(i)$ for some $\eta'(i)$ going to zero with i. Call the fiber passing

through p by $\mathcal{O}_p$.

By the argument in [31, Appendix 1] we can find (for large enough i) a constant δ and a fiber $\mathcal{O}_{q_i} \subset O_i$ with $d(q_i, p_i) < \delta$, such that the orbit $\mathcal{O}_{q_i}$ has normal injectivity radius at least 20δ. One can also bound the second fundamental form of $\mathcal{O}_{q_i}$. The normal bundle has a natural metric inherited from g'_η, and a locally flat connection inherited from the nilpotent action on the fiber $\mathcal{O}_{q_i}$. By the second fundamental form bounds, for i sufficiently large, the restricted metric g'_η makes $\mathcal{O}_{q_i}$ a ϵ-quasiflat orbifold. Further, the metric induced by the locally flat connection makes the normal bundle to $\mathcal{O}_{q_i}$ a locally flat Riemannian orbivector bundle over an ϵ-quasiflat base. Denote by g''_η the metric tensor which is induced on this normal bundle by the locally flat connection.

Denote by $N_{10\delta}(\mathcal{O}_{q_i})$ the vectors in the normal bundle of the fiber $\mathcal{O}_{q_i}$ with length at most 10δ. One can show, for i sufficiently large, that the normal-exponential map $f \colon N_{10\delta}(\mathcal{O}_{q_i}) \to O_i$ induces a map onto the 10δ-tubular neighborhood (with respect to g'_η of O_i) with $||f^*g - g''_\eta||_g < \eta''(i)$ for some $\eta''(i)$ tending to zero with i. Further, the image of f contains $B_p(\delta)$ and we can consider the restricted map $f^{-1} \colon B_p(\delta) \to N_{10\delta}(\mathcal{O}_{q_i})$. This gives a metric $f^{-1*}(g''_\eta) = g'$ on $B_p(\delta)$ with the desired properties. The model space O' is the normal bundle of $\mathcal{O}_{q_i}$ with the metric g''_η and the isometry of $B_p(\delta)$ is given by f. However, we assumed that no such structure could exist, and thus the statement follows by contradiction.

$\square$

2.5.2 Diameter estimate

Lemma 2.5.4. *Let $M = N/\Gamma$ be a n-dimensional $\epsilon(n)$-quasiflat orbifold and $\Lambda = N \cap \Gamma < \Gamma$ the translational normal subgroup of Γ. Further, let γ_i be a short basis for M as in Lemma 2.3.23. Then we have* Diam $(M) \leq C(n) \max_{i=1,\ldots,n} |\gamma_i|$.

Here, we use the notation $|\gamma| = |\gamma|_e = d(\gamma, e)$, where e is the identity element of N, and d is the Riemannian distance function on N.

Proof: Denote $D = \max_{i=1,\ldots,n} |\gamma_i|$. To prove the lemma we can assume $\Gamma = \Lambda$, because taking a finite cover will only increase the diameter. We will show a more general statement. The statement we prove is that, if a cocompact lattice $\Lambda < N$ admits a triangular basis $\gamma_1, \ldots, \gamma_n$ in the sense of definition 2.3.21, then Diam $(M) \leq C(n) \max_{i=1,\ldots,n} |\gamma_i|$. The statement is obvious for $n = 1$, and we will proceed to assume that $n > 1$ and that the statement has been shown for $n - 1$-dimensional simply connected nilpotent Lie groups.

Assume that $\Lambda \subset N$ is a discrete co-compact lattice of a n-dimensional nilpotent Lie group with a triangular basis γ_i and set $\gamma_i = e^{X_i}$ with $X_i \in T_e N$ a basis and $[X_i, X_j] \in \mathrm{span}(X_1, \ldots, X_{i-1})$ for $i < j$. Since γ_i is assumed to be a basis, the vectors X_i form a basis for $T_e N$. (Recall Definition 2.3.21). In particular, we can introduce coordinates $(t_1, \ldots, t_n) \to \prod_{i=1}^{n} e^{t_{n-i} X_{n-i}}$, such that the product can be expressed as

$$\prod_{i=1}^{n} e^{t_{n-i} X_{n-i}} \times \prod_{i=1}^{n} e^{s_{n-i} X_{n-i}} = \prod_{i=1}^{n} e^{(s_{n-i} + t_{n-i} + p_{n-i}(s,t)) X_i},$$

where $p_i(s, t)$ depends only on s_j, t_j for $j < i$. We will show that Diam $(N) \leq C(n)D$.

Since γ_1 is in the center of N, we have that X_1 commutes with all left-invariant

fields. Further, by the Koszul formula $t \to e^{tX_1}$ is seen to be a geodesic in N. Thus, $d(e^{sX_1}, e) = sd(e^{X_1}, e) = s|\gamma_1|$. Consider the Lie group $N' = N/L$, where $L = \{e^{tX_1}\}$, and a discrete subgroup $\Gamma' = \langle \gamma_i', i = 1, \dots, n-1 \rangle = \Gamma/\langle \gamma_1^n, n \in \mathbb{Z} \rangle$. Here, the generators correspond to equivalence classes of generators of Γ in $\Gamma/\langle \gamma_1^n, n \in \mathbb{Z} \rangle$: $\gamma_i' = [\gamma_{i+1}]$, and if we denote $e^{X_i'} = \gamma_i'$, we have co-ordinates $(t_1, \dots, t_{n-1}) \to \prod_{i=1}^{n-1} e^{t_{n-i} X_{n-i}'}$. Here the exponential map is in N'. Take as the metric on N' the quotient metric, which is induced by the restricted Riemannian metric on a vector space perpendicular to L. With respect to this metric $|\gamma_i'| \le |\gamma_{i+1}| \le |\gamma_n| \le D$ for all $1 \le i \le n-1$. The norm $|\cdot|$ on N' is defined as $|n'| = d(n'e', e')$, where $n' \in N'$ and e' is the identity element of N'.

Define $M' = N'/\Gamma'$. The group L acts naturally on M by isometries and M/L is isometric to M'. Further, the L-orbits in M have length $|\gamma_1|$, so by Lemma 2.3.7 $d_{GH}(M, M') \le 3|\gamma_1|$. By the inductive hypothesis, Diam $(M') \le C(n-1)D$. Thus,

$$
\begin{aligned}
\text{Diam } (M) &\le \text{Diam } (M') + 2d_{GH}(M, M') \le C(n-1)D + 6D \\
&\le (6 + C(n-1))D.
\end{aligned}
$$

In particular, we can choose $C(n) = 6n$.

$\square$

Lemma 2.5.5. *Let N^n be a simply connected nilpotent Lie group and $M^k \lhd N^n$ a connected normal subgroup thereof. Assume that N is equipped with a left-invariant metric which satisfies the sectional curvature bound $|K| \le 2$. Then*

for some universal $\delta(n) > 0$ and the $4\delta(n)$-tubular neighborhood $N_{4\delta(n)}(M^k)$ of M^k in N^n there is a map $F\colon N_{\delta(n)}(M^k) \to M^k \times \mathbb{R}^{n-k}$ whose restriction onto a ball $F\colon B_e(2\delta(n)) \to M^k \times \mathbb{R}^{n-k}$ is $A(n)$-bi-Lipschitz. Also, F is an isometry when restricted to $F^{-1}(M^k \times \{0\})$. Furthermore, for any affine isometry I of N^n which preserves M^k, also $F \circ I \circ F^{-1}$ is a restriced affine isometry of $M^k \times \mathbb{R}^{n-k}$.

Proof: Introduce an orthonormal triangular basis $X_1, \ldots, X_k$ for M^k, and extend it to an orthonormal basis for N^n with vectors $X_{k+1}, \ldots, X_n$. Consider the map $G\colon M^k \times \mathbb{R}^{n-k} \to N$, which is defined by

$$G(m, x) = m e^{\sum_{i=k+1}^n x_i X_i}.$$

This is smooth. The exponential map used here is the Lie group exponential. Clearly, any affine isometry I preserving M^k will conjugate to be an isometry of $M^k \times \mathbb{R}^{n-k}$ by considering its action on M^k and its normal bundle. Apply the Campbell-Hausdorff formulas to compute the quantities $\partial_{x_i} G$.

$$\partial_{x_i} G(m, x) = X_i + \sum_{k=1}^n \sum_{i_1, \ldots, i_k} M^k_{i_1, \ldots, i_k} x_{i_1} \cdot x_{i_2} \cdots x_{i_k} [X_{i_1}, [X_{i_2}, [\ldots, [X_{i_k}, X_i]]]].$$

Denote the error terms in the sum by $Y_i = \partial_{x_i} G(m, x) - X_i$. Now assume $x_i \leq \delta$ for any i. The metric tensor can now be expressed in the given co-ordinates as follows.

$$\begin{aligned}
\langle \partial_{x_i} G(m, x), \partial_{x_j} G(m, x) \rangle &= \langle X_i + Y_i, X_j + Y_j \rangle \\
&= g_{ij} + \langle Y_i, Y_j \rangle + \langle X_i, Y_j \rangle + \langle Y_i, X_j \rangle
\end{aligned}$$

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Since $||Ad|| \leq C'(n)$, we get $||Y_i|| \leq M(n)\delta^k C'(n)^k$. Here, δ_{ij} is the Dirac delta. Choose $\delta(n) < \frac{1}{10^3 M(n) C'(n)}$ and apply Cauchy's inequality to give

$$1 - \frac{3}{8} \leq \langle \partial_{x_i} G(m, x), \partial_{x_i} G(m, x) \rangle \leq 1 + \frac{3}{8}, \tag{2.5.6}$$

whenever $x_i \leq 10\delta(n)$. Let $C(\delta(n)) = \{(m, x) | x_i \leq 10\delta(n)\} \subset M^k \times \mathbb{R}^{n-k}$. From its definition and (2.5.6), we can see that $G(C(\delta(n))$ contains the $4\delta(n)$-tubular neighborhood $N_{4\delta(n)}(M^k)$ of M^k. Also, when restricting G^{-1} onto $B_e(2\delta(n)) \subset N$ we get a $\frac{1+\frac{3}{8}}{1-\frac{3}{8}}$-bi-Lipschitz into $M^k \times \mathbb{R}^{n-k}$. Thus, the inverse G^{-1} is our desired map. We remark, that the distortion can be made arbitrarily small by choosing a smaller $\delta(n)$.

$\square$

Chapter 3

Quantitative connectivity and Poincaré inequalities

3.1 Introduction

3.1.1 Overview

This chapter involves two main contributions: a new quantitative connectivity condition that characterizes Poincaré inequalities and applying it to various contexts. These aspects of the theory involve different technical challenges, and somewhat distinct historical backgrounds. Thus, in the following presentation these topics are presented in separate subsections. For general exposition of related ideas and concepts, see the expository articles $[16, 72, 88, 132]$.

All of the results in this chapter will be stated for proper complete metric measure spaces (X, d, μ) equipped with a Radon measure μ. Most of the results could also be modified to apply to non-complete spaces.

3.1.2   Characterizing spaces with Poincaré inequalities

Poincaré inequalities and doubling measures are a useful tool in analysis of differentiable manifolds and metric spaces alike (see Section 3.2 for the definitions). Following the convention established by Cheeger and Kleiner a space with both a doubling measure and a Poincaré inequality is referred to as a *PI-space* or *p*-Poincaré spaces (see Definition 3.2.5 below). PI-spaces possess an expansive theory of first order analysis including a differentiation theory [28, 33] and a theory of quasiconformal maps [74]. A detailed theory of Sobolev spaces on such spaces with many familiar properties was developed by Cheeger [28]. For a general overview and some later developments, see the beautiful expository article by Heinonen [72] and the book [75].

Due to the many desirable structural properties that PI-spaces have, much effort has been expended to understand what geometric properties guarantee a Poincaré inequality in some form. By now several classes of spaces with Poincaré inequalities are known. See for example [20, 29, 35, 78, 84, 92, 101, 119, 127, 128]. The proofs that these examples satisfy Poincaré inequalities are often challenging, and make extensive use of the geometry of the underlying space. Our main theorem is thus motivated by the following question, which has also been posed in [72].

Question 3.1.1. Which geometric properties characterize PI-spaces? Find new and weak conditions on a space that guarantee a Poincaré inequalities.

As an answer to the above question we introduce the following connectivity, or avoidance, property for a general metric measure space. This condition is very similar to the condition appearing in the work of Sean Li and David Bate [12, Theorem 3.8]. The main difference is that Li and Bate treat this property only in connection

to certain classes of differentiability spaces, while here we isolate it as a property of a general metric measure space (see below for more discussion). Another related condition appears in connection to ∞-Poincaré inequality discussed in [51, Theorem 3.1(f)], but there one must assume $\mu(E){=}0$.

Definition 3.1.2. Let[1] $0 < \delta$, $0 < \epsilon < 1$ and $C > 1$ be given. If $(x, y) \in X \times X$ is a pair of points with $d(x, y) = r > 0$, then we say that the pair (x, y) is (C, δ, ϵ)-connected if for every Borel set E such that $\mu(E \cap B(x, Cr)) < \epsilon\mu(B(x, Cr))$ there exists a 1-Lipschitz[2] curve fragment $\gamma \colon K \to X$ connecting x and y, such that the following hold.

1. $\mathrm{len}(\gamma) \leq Cd(x, y)$

2. $|\mathrm{Undef}(\gamma)| < \delta d(x, y)$

3. $\gamma^{-1}(E) \subset \{0, \max(K)\}$

We call (X, d, μ) a $(C, \delta, \epsilon, r_0)$-connected space, if every pair of points $(x, y) \in X$ with $r = d(x, y) \leq r_0$ is (C, δ, ϵ)-connected.

We say that X is uniformly $(C, \delta, \epsilon, r_0)$-connected along S, if every $(x, y) \in X$ with $x \in S$ and with $d(x, y) \leq r_0$ is (C, δ, ϵ)-connected in X.

If (X, d, μ) is $(C, \delta, \epsilon, r_0)$-connected for all r_0, we simply say that (X, d, μ) is (C, δ, ϵ)-connected.

The notion of a curve fragment and related terminology is introduced in Definition 3.2.6 below. A related version of this condition with curves is the following.

---

[1]The definition is only interesting for $\delta < 1$. However, we allow it to be larger to simplify some arguments below. If $\delta \geq 1$ then the curve fragment could consist of two points $\{0, d(x, y)\}$, and $\gamma(0) = x, \gamma(d(x, y)) = y$.

[2]According to Lemma 3.2.7 we could simply assume some Lipschitz bound, but it is easier to normalize the situation.

Definition 3.1.3. Let $0 < \delta$, $0 < \epsilon < 1$ and $C > 1$ be given. If $(x, y) \in X \times X$ is a pair of points with $d(x, y) = r > 0$, then we say that the pair (x, y) is (C, δ, ϵ)-connected by a curve if for every Borel set E such that $\mu(E \cap B(x, Cr)) < \epsilon \mu(B(x, Cr))$ there exists a 1-Lipschitz curve $\gamma \colon I \to X$ connecting x and y, such that the following hold.

1. $\text{len}(\gamma) \leq C d(x, y)$

2. $|\gamma^{-1}(E)| < \delta d(x, y)$

Similarly we say that (X, d, μ) is $(C, \delta, \epsilon, r_0)$-connected by curves, if every pair of points $(x, y) \in X$ with $r = d(x, y) \leq r_0$ is (C, δ, ϵ)-connected by a curve.

If (X, d, μ) is $(C, \delta, \epsilon, r_0)$-connected by curves for all r_0, we simply say that (X, d, μ) is (C, δ, ϵ)-connected by curves.

Below in Lemma 3.3.10, we see that these two definitions are roughly equivalent. However, while less intuitive to state with curve fragments, that language is necessary for the application to differentiability in the final Chapter 4. Also, it is often easier to apply in practice.

Our connectivity condition is formally very similar to the definition of Muckenhoupt weights A_∞ in terms of a quantitative version of absolute continuity. For definitions and a beautiful exposition of this theory consult [135, Section V]. Consequently, many of our arguments and theorems are motivated by the corresponding proofs for Muckenhoupt weights. A major difference is that the connectivity property, and its finer variant 3.1.5, does not self-improve in the same way as the Muckenhoupt-condition.

We present some general results for this new notion. For example, it is stable under Gromov-Hausdorff limits.

Theorem 3.1.4. *If* $(X_i, d_{,i}, \mu_i, x_i) \to (X, d, \mu, x)$ *is a convergent sequence of proper pointed metric measure spaces in the measured Gromov-Hausdorff sense, and each* X_i *is* (C, δ, ϵ)*-connected, then the limit space* (X, d, μ) *is also* $(C, 2\delta, \frac{\epsilon}{2})$*–connected.*

In terms of this new notion of connectivity we prove the following.

Theorem 1.2.38. *A* (D, r_0)*-doubling complete metric measure space* (X, d, μ) *admits a local* $(1, p)$*-Poincaré inequality for some* $1 \leq p < \infty$ *if and only if it is locally* (C, δ, ϵ)*-connected for some* $0 < \delta, \epsilon < 1$*. Both directions of the theorem are quantitative in the respective parameters.*

Note, the doubling is not needed for the converse statement, because by Lemma 3.3.2 it is implied by this condition. Similarly, the theorems below could dispense with the assumption of D-doubling. However, we add this to explicate the dependence of the parameters on the implied doubling constant.

The previous theorem fully characterizes PI-spaces. Another similar characterization of spaces with Poincaré inequalities appears in the work of Keith [84], but there the characterization is for a fixed p, and in terms of modulus estimates. In some cases, one might be interested in Poincaré inequalities without knowing *a priori* the value of p sought. Also, in general p might be arbitrarily large [128]. In such contexts, our characterization seems easier to apply.

Other characterizations in terms of Loewner properties appeared earlier in the paper of Heinonen and Koskela [74], but there one had to assume Ahlfors regularity. Compared to these results, our result is *a priori* weaker in that it does not specify the optimal range of p. However, it leads to new applications because the condition is easier to obtain in some contexts, and we don't need Ahlfors regularity. This is

demonstrated in the results below, where we apply this theorem to prove Poincaré inequalities for new classes of spaces. We remark that Keith's characterization also has several interesting applications [101, 128] and our theorem doesn't replace it.

We introduce a finer notion of connectivity in terms of which the main theorem becomes easier to state. It should be compared to the relationship between A_∞-weights and A_p-weights in classical analysis.

Definition 3.1.5. We call a metric measure space (X, d, μ) finely α-connected with parameters (C_1, C_2) if for any $0 < \tau < 1$ the space is $(C_1, C_2\tau^\alpha, \tau)$-connected. Further, we say that the space is locally finely (α, r_0)-connected with parameters (C_1, C_2) if it is $(C_1, C_2\tau^\alpha, \tau, r_0)$-connected for any $0 < \tau < 1$. If we do not explicate the dependence on the constants, we simply say (X, d, μ) is finely α-connected if constants C_1, C_2 exist with the previous properties. Similarly we define locally finely (α, r_0)-connected spaces.

For example, note that $\mathbb{R}^n$ is finely $\frac{1}{n}$-connected. Similarly, the space arising from gluing two copies of $\mathbb{R}^n$ along the origin is also $\frac{1}{n}$-connected. If the Lebesgue measure λ on $\mathbb{R}^n$ is deformed by an A_p-Muckenhoupt weight w, then the space $(\mathbb{R}^n, |\cdot|, wd\lambda)$ is $\frac{1}{pn}$-connected. This follows from the results in [135, Chapter V]. See also the calculations in the introduction to this thesis in Chapter 1.

The finer notion of connectivity is used as an intermediate step to prove Theorem 3.3.17. It also allows us to better control the exponent in the Poincaré inequality. The connection between the previous two definitions is given by the following theorem.

Theorem 1.2.40. *Assume $0 < \delta < 1$ and $0 < \epsilon, r_0, D$. If (X, d, μ) is a $(C, \delta, \epsilon, r_0)$-connected (D, r_0)-doubling metric measure space, then there exist $0 < \alpha, C_1, C_2$*

such that it is also finely $(\alpha, r_0/(C_1 8))$-connected with parameters (C_1, C_2). We can choose $\alpha = \frac{\ln(\delta)}{\ln(\frac{\epsilon}{2M})}$, $C_1 = \frac{C}{1-\delta}$ and $C_2 = \frac{2M}{\epsilon}$, where $M = 2\max(D^{-\log_2(1-\delta)+1}, D^3)$.

Remark: We remark, that while the connectivity conditions are very similar to Muckenhoupt conditions, a major difference holds. Both conditions admit a "self-improvement" property of the form in Theorem 1.2.40. This can be thought of as the statement that A_∞-weights are A_p-weights for some p. However, Muckehoupt weights and Poincaré inequalities also possess a different type of self-improvement. An A_p-weight automatically belongs to $A_{p-\epsilon}$ for some $\epsilon > 0$, and a $(1,p)$-Poincaré inequality on a doubling metric measure space improves to a $(1, p - \epsilon)$-Poincaré inequality (see [86]). However, a finely α-connected space may fail to be $\alpha + \epsilon$-connected for any ϵ. For example, $\mathbb{R}^n$ is finely $\frac{1}{n}$-connected but not finely α-connected for any $\alpha > \frac{1}{n}$. Thus, fine connectivity is not an open condition.

Theorem 1.2.41. *Let (X, d, μ) be a D-measure doubling locally finely (α, r_0)-connected metric measure space, then for any $p > \frac{1}{\alpha}$ there exists C_{PI}, C_1 such that the space satisfies a local $(1, p, r_0/(8C_1), C_{PI}, 2C_1)$-Poincaré inequality. In short the space is a PI-space.*

We can set $M = 2(D^4)^{\frac{1}{p\alpha-1}}$, $\delta = \left(\frac{D^4}{M^p}\right)^\alpha$, $C_1 = \frac{C}{1-\delta}$, $C_3 = \frac{C_1 M}{1-M\delta}$ and $C_{PI} = 2C_3$.

The range of p's in this theorem is tight in general. Take the space Y arising as the gluing of two copies of $\mathbb{R}^n$ through their origins. The resulting space (Y, d, μ), where d is the glued metric and μ the sum of the measures on each component, is $\frac{1}{n}$-connected and satisfies a $(1, p)$-Poincaré inequality only for $p > n$. On the other hand, for some particular examples, such as $\mathbb{R}^n$ or Ricci-bounded manifolds, we know that the space satisfies a $(1, 1)$-Poincaré inequality, but are only $\frac{1}{n}$-connected. Also, if this result is applied to Muckenhoupt weights $w \in A_q(\mathbb{R}^n)$, we would

get a Poincaré inequality for $p > nq$, while it is known to hold for $p > q$ (see e.g. [135, Chapter V] or [54]). Finally, we remark that this theorem is not an equivalence. A $(1, p)$-PI-space might not be α-connected for any $\alpha \geq \frac{1}{p}$.

The proofs of the three main theorems are all very similar. They are based on an iterated gap-filling and limiting argument. Recall, that the curve fragments guaranteed by Definitions 1.2.32 and 1.2.39 can contain gaps. However, the core idea is to re-use the connectivity condition at the scale of these gaps and to replace them with finer curve fragments. A similar argument appears in [12, Lemma 3.6], and earlier in [74]. This procedure can be used quite easily to prove quasiconvexity. In that case, we can always set the obstacle $E = \emptyset$ when we apply the connectivity definition. However, for the application to Theorems 1.2.40 and 1.2.41, we will need to define obstacles at different scales. To do this successfully, we use a maximal function estimate. Such iterative arguments employing maximal functions are standard in harmonic analysis, and especially the proof of Theorem 1.2.41 resembles the proofs of Theorems V.3.1.4 and V.5.1.3 in [135, Section V]).

Finally, while our condition and conclusion do not use the modulus estimates of Keith [84] and Heinonen-Koskela [74], it is not surprising that at the end we are able to connect our condition to a certain type of modulus estimate. To state it, we need the following terminology. Define the set of curves

$$\Gamma_{x,y,C} = \{\gamma\colon [0, 1] \to X | \gamma(0) = x, \gamma(1) = y, \operatorname{len}(\gamma) \leq Cd(x, y)\}.$$

We call a non-negative measurable function ρ admissible for the family $\Gamma_{x,y,C}$ if for any rectifiable $\gamma \in \Gamma_{x,y,C}$ we have

$$\int_\gamma \rho \, ds \geq 1.$$

The p-modulus of the curve family (centered at x, and scale $s > d(x,y)$) is then defined as

$$\mathrm{Mod}_p^{x,s}(\Gamma_{x,y,C}) = \inf_{\rho \text{ is admissible for } \Gamma_{x,y,C}} \fint_{B(x,s)} \rho^p \, d\mu.$$

Theorem 3.1.6. *If (X, d, μ) is a complete (D, r_0)-measure doubling metric measure space which is also locally finely (α, r_0)-connected with parameters (C_1, C_2), then for any $x, y \in X$ with $d(x, y) = r < r_0/(8C_1)$, any $p > \frac{1}{\alpha}$ we have*

$$\mathrm{Mod}_p^{x,2C_1 r}(\Gamma_{x,y,C_3}) \geq \frac{C_3^p}{r^{p-1}},$$

where C_3 is as in Lemma 3.3.11.

3.1.3 Applications of the Characterization

Our new connectivity assumption in Definition 3.1.2 can be used in a variety of contexts to establish true Poincaré inequalities under *a priori* weaker connectivity properties.

The first result concerns metric spaces with weak Ricci bounds. These spaces originally arose following the work of Cheeger and Colding on Ricci limits [29, 30]. Many different definitions appeared such as different definitions for $CD(K, n)$ [97, 136, 137], $CD^*(K, n)$ [10] and a strengthening $RCD(K, n)$ [2, 4]. Ohta also defined a very weak form of a Ricci bound by the *measure contraction property* and introduced $MCP(K, N)$-spaces [113]. Spaces satisfying one of the stronger

conditions $(CD(K, N)$, $RCD(K, N)$ or $RCD(K, N)^*$ were all known to admit Poincaré inequalities [118]. It was also known that a non-branching $MCP(K, N)$-space would admit a $(1, 1)$-Poincaré inequality. This was observed by Renesse [146], whose proof was essentially a repetition of classical arguments in [29]. However, Rajala had conjectured that this assumption of non-branching was inessential. $MCP(K, n)$ spaces are interesting partly because they are known to include some very non-Euclidean geometries such as the Heisenberg group and certain Carnot-groups [79, 121]. We are able to prove the following.

Theorem 1.2.42. *If (X, d, μ) is a $MCP(K, n)$ space, then it satisfies a local $(1, p)$-Poincaré-inequality for $p > n + 1$. Further, if $K \geq 0$, then it satisfies a global $(1, p)$-Poincaré inequality.*

We are left with the following open problem.

Open Question: Does every $MCP(K, n)$-space admit a local $(1, 1)$-Poincaré inequality?

We next consider more general self-improvement phenomena for Poincaré inequalities. In a celebrated paper, Keith and Zhong proved that in a doubling complete metric measure space a $(1, p)$-Poincaré-inequality immediately improves to a $(1, p - \epsilon)$-Poincaré inequality [86]. We ask if such self-improvement can be pushed to show that more general Poincaré-type inequalities also self-improve.

By a Poincaré type-inequality we will refer to inequalities that control the oscillation of a function by some, possibly non-linear, functional of its gradient. Such inequalities have appeared in the work of Semmes [130], Bate and Li [12] and Jana Björn in [15]. We will here show that on geodesic doubling metric measure spaces many types of Poincaré inequalities imply a true Poincaré inequality. In particu-

lar, most of the definitions of Poincaré inequalities produce the same category of PI-spaces. Consider the following definition.

Definition 3.1.7. Let (X, d, μ) be a metric measure space and $f \colon X \to \mathbb{R}$ a Lipschitz function. We call a non-negative Borel-measurable g an upper gradient for f if for every rectifiable curve $\gamma \colon [0, L] \to X$ we have

$$|f(\gamma(0)) - f(\gamma(L))| \leq \int_0^L g(\gamma(t)) ds_\gamma.$$

With this definition, we can define a non-homogeneous Poincare inequality.

Definition 3.1.8. Let (X, d, μ) be a metric measure space. Let $\Phi, \Psi \colon [0, \infty) \to [0, \infty)$ be increasing functions with the following properties.

- $\lim_{t \to 0} \Phi(t) = \Phi(0) = 0$

- $\lim_{t \to 0} \Psi(t) = \Psi(0) = 0$

We say that (X, d, μ) satisfies a non-homogeneous (Φ, Ψ, C)-Poincaré inequality if for every 2-Lipschitz[3] function $f \colon X \to \mathbb{R}$, every upper gradient $g \colon X \to \mathbb{R}$ and every ball $B(x, r) \subset X$ we have

$$\fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq r \Psi \left(\fint_{B(x,Cr)} \Phi \circ g \, d\mu \right).$$

This class of Poincaré inequalities subsumes the ones considered by Björn in [15] and the Non-homogeneous Poincaré Inequalities (NPI) considered by Bate and Li in [12]. We get the following theorem and corollary.

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[3]The constant 2 is only used to simplify arguments below. Any fixed bound could be used. Also, by adjusting the right-hand side by scaling with the Lipschitz constant of f, the inequality could be made homogeneous.

Theorem 1.2.44. *Suppose that (X, d, μ) is a quasiconvex D-measure doubling metric measure space and satisfies a non-homogeneous (Φ, Ψ, C)-Poincaré inequality. Then (X, d, μ) has a true $(1, q)$-Poincaré inequality for some $q > 1$. All the variables are quantitative in the parameters. In particular, a non-homogeneous Poincaré inequality in the sense of [12] implies a true Poincaré inequality.*

Remark: One could weaken the assumption of quasiconvexity if we allow any function f instead of Lipschitz functions, or by modifying the inequality to the ones considered in [12, 50]. We also obtain the following result strengthening the conclusion of Dejarnette [47].

Theorem 1.2.45. *Suppose (X, d, μ) is a doubling metric measure space satisfying a weak $(1, \Phi)$-Orlicz-Poincaré inequality in the sense of [15], then it satisfies also a $(1, q)$-Poincaré inequality for some q.*

Finally, we will show a theorem concerning A_∞-weights on metric measure spaces. This generalizes some aspects from [56] concerning sub-Riemannian metrics and vector fields satisfying the Hörmander condition. For the definition, see section 3.4 or Definition 1.2.33.

Theorem 1.2.49. *Let (X, d, μ) be a geodesic D-measure doubling metric measure space. If (X, d, μ) satisfies a $(1, p)$-Poincaré inequality and $\nu \in A_\infty(\mu)$, then there is some $q > 1$ such that (X, d, ν)-satisfies a $(1, q)$-Poincaré inequality.*

Sometimes Sobolev inequalities would be more desired, and such inequalities could be proven using techniques from [68].

3.1.4 Structure of chapter

We first cover some general terminology and frequently used lemmas. In the third section, we introduce our notion of connectivity and prove basic properties and finally derive Poincaré inequalities. In the fourth section, we apply the results in both new and classical settings.

3.2 Notational conventions and preliminary results

Recall, we are studying the geometry of complete proper metric measure spaces (X, d, μ). Where not explicitly stated, all the measures considered in this chapter will be Radon measures. Recall the definition of a Poincaré-inequality.

Definition 3.2.1. Let $p \in [1, \infty)$, $C, C' > 0$ be constants. We say that a proper metric measure space (X, d, μ) equipped with a Radon measure satisfies a $(1, p)$-Poincaré inequality with constants (C, C') if for every $r > 0$, every $x \in X$ and every Lipschitz function $f \colon X \to \mathbb{R}^n$.

$$\fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq Cr \left(\fint_{B(x,C'r)} \operatorname{lip} f^p \, d\mu \right)^{\frac{1}{p}}. \tag{3.2.2}$$

Additionally, we say that the metric measure space (X, d, μ) satisfies a local $((1, p, r_0)-)$Poincaré inequality if for some $r_0 > 0$ the previous holds for all $r < r_0$. A space satisfies a local $(1, p)$-Poincaré inequality if the aforementioned holds for some $r_0 > 0$.

Remark: There are different versions of Poincaré inequalities and their equivalence in various contexts has been studied in [84] and [68, 70]. The quantity $\operatorname{lip} f$

on the right hand side could also be replaces by Lip f (see equations 3.2.4 and 3.2.3 below), and on complete spaces by an upper gradient (in any of the senses discussed in [28,50,74]). For non-complete spaces, which we do not focus on, the issue is slightly more delicate (see counterexamples in [90]), but can often be avoided by taking completions. Further, as long as the space is complete, we do not need to constrain the inequality for Lipschitz functions but could also use appropriately defined Sobolev spaces. Note that on the right hand side the ball is enlarged by a factor C^4. On geodesic metric spaces the inequality can be improved to have $C = 1$ [68].

Recall the definitions of the local Lipschitz constants

$$\text{Lip } f(x) = \limsup_{r \to 0} \sup_{y \in B(x,r)} \frac{|f(x) - f(y)|}{r} \tag{3.2.3}$$

and

$$\text{lip } f(x) = \liminf_{r \to 0} \sup_{y \in B(x,r)} \frac{|f(x) - f(y)|}{d(x,y)}. \tag{3.2.4}$$

We use the following notion of PI-space.

Definition 3.2.5. A proper metric measure space (X, d, μ) equipped with a Radon measure μ is called a $(1, p, D, r_0, C_{PI}, C)$-PI space if it is (D, r_0)-doubling and satisfies a local $(1, p, r_0)$-Poincaré inequality with constants (C, C_{PI}). Further a space is called a $(1, p, D, r_0)$-PI space, a $(1, p, r_0)$-PI space or simply PI-space if there exists remaining constants so that the space satisfies the aforementioned property.

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[4]Some authors refer inequalities with $C > 1$ as weak Poincaré inequalities, but we do not distinguish between these different terms.

Definition 3.2.6. A curve fragment in a metric space (X, d) is a Lipschitz map $\gamma\colon K \to X$, where $K \subset \mathbb{R}$ is compact. For simplicity, we translate the set so that $\min(K) = 0$. We say the curve fragment connects points x and y if $\gamma(0) = x, \gamma(\max(K)) = y$. Further define $\mathrm{Undef}(\gamma) = [0, \max(K)] \backslash K$. If $K = [0, \max(K)]$ is an interval we simply call γ a curve.

The length of a curve fragment is defined as

$$\mathrm{len}(\gamma) = \sup_{x_1, \ldots, x_n \in K} \sum_{i=1}^{n} d(\gamma(x_{i+1}), \gamma(x_i)).$$

Since γ is assumed to be Lipschitz we have $\mathrm{len}(\gamma) \leq \mathrm{LIP}\,(\gamma) \max(K)$.

Analogous to curves we can define an integral over a curve fragment $\gamma\colon K \to X$. Denote $\sigma(t) = \sup_{x_1 \leq \cdots \leq x_n \in K \cap [0,t]} \sum_{i=1}^{n} d(\gamma(x_{i+1}), \gamma(x_i))$. The function $\sigma|_K$ is Lipschitz on K. Thus, it is differentiable for almost every density point $t \in K$ and for such t we set $d_\gamma(t) = \sigma'(t)$ and call it the *metric derivative* (see [3] and [11]). We define an integral of a Borel function g as follows

$$\int_\gamma g \; ds = \int_K g(\gamma(t)) \cdot d_\gamma(t) \; dt,$$

when the right hand side makes sense. This is true for example if g is bounded from below or above.

We will need to do some technical modifications of domains of Lipschitz functions and as such we formulate two lemmas useful for that. First, we give a standard re-parametrization result which generalizes to curve fragment the unit-speed parametrization for curves.

Lemma 3.2.7. *Let (X, d) be a complete metric space, $x, y \in X$ points and $\gamma\colon K \to X$ a curve fragment connecting x to y. There is a compact set $K' \subset [0, \mathrm{len}(\gamma)]$,*

with $0, \mathrm{len}(\gamma) \in K'$, and continuous function $\phi\colon K' \to K$ such that $\gamma' = \gamma \circ \phi$ is a 1-Lipschitz curve fragment connecting x to y. Further, the metric derivative is given, almost everywhere, by $d_{\gamma'} = d_\gamma \circ \phi$. If $(a_i, b_i) \subset [0, \mathrm{len}(\gamma)] \setminus K'$ is a maximal open interval in $[0, \mathrm{len}(\gamma)] \setminus K'$, we have $d(\gamma(a_i), \gamma(b_i)) = |a_i - b_i|$, and

$$\mathrm{len}(\gamma') = \mathrm{len}(\gamma).$$

Proof: Define $\sigma\colon K \to \mathbb{R}$ by $\sigma(t) = \sup_{x_i \in K \cap [0,t]} \sum_{i=1}^n d(\gamma(x_{i+1}), \gamma(x_i))$, where $x_i \leq x_j$ for $i \leq x_j$. The function $\sigma|_K$ is automatically continuous and increasing and the result follows by considering $\sigma(K) = K'$ and $\phi = \sigma^{-1}$. The remaining properties are easy to check.

$\square$

Sometimes gaps in a curve will need to be enlarged.

Lemma 3.2.8. *Let (X, d) be a complete metric space, $x, y \in X$ points and $\gamma\colon K \to X$ a 1-Lipschitz curve fragment. If $[0, \max(K)] \setminus = \bigcup_i (a_i, b_i)$ where $(a_i, b_i) = I_i$ are disjoint open intervals, and $C > C_i > 1$ are constants, then there exists a compact K', a continuous and increasing function $\phi\colon [0, \max(K)] \to [0, \max(K')]$ such that $\phi(K) = K'$ and $\gamma' = \gamma \circ \phi^{-1}$ defines a 1-Lipschitz curve fragment on K'. Also $[0, \max(K')] \setminus K' = \bigcup_i (\phi(a_i), \phi(b_i))$, $|\phi(a_i) - \phi(b_i)| = C_i |a_i - b_i|$ and*

$$\max(K') \leq \max(K) + (C - 1)|\mathrm{Undef}(\gamma)|.$$

Proof: Define $\phi(t) = \int_0^t (1_K + \sum_i C_i 1_{I_i})\, dt$. The rest of the estimates are easy to see.

For a locally integrable function $f\colon X \to \mathbb{R}$ and a ball $B = B(x, r) \subset X$ with $\mu(B(x, r)) > 0$ we define the average of a function as

$$f_B = \fint_B f \, d\mu = \frac{1}{\mu(B(x, r))} \int f(y) \, d\mu_y.$$

We will frequently use the Hardy-Littlewood maximal function at scale $s > 0$:

$$M_s f(x) = \sup_{x \in B = B(y,r), r < s} \fint_B |f|(x) \, d\mu,$$

and refer to [135] for a standard proof of the following result.

Theorem 3.2.9. *Let (X, d, μ) be a D-measure doubling metric measure space and $s > 0$ and $B(x, r) \subset X$ arbitrary, then for any non-negative $f \in L^1$ and $\lambda > 0$ we have*

$$\mu\left(\{M_s f > \lambda\} \cap B(x, r)\right) \leq D^3 \frac{\|f 1_{B(x, r+s)}\|_{L^1}}{\lambda}$$

and thus for any $1 < p \leq \infty$

$$\|M_s f\|_{L^p(\mu)} \leq C(D, p) \|f\|_{L^p(\mu)}.$$

Definition 3.2.10. A metric space (X, d) is called L-quasiconvex if for every $x, y \in X$ there exists a Lipschitz curve $\gamma\colon [0, 1] \to X$ such that $\gamma(0) = x, \gamma(1) = y$ and $\mathrm{len}(\gamma) \leq L d(x, y)$. A metric space is locally (L, r_0)-quasiconvex if the same holds for all $x, y \in X$ with $d(x, y) \leq r_0$.

Definition 3.2.11. A metric space (X, d) is called geodesic if for every $x, y \in X$ there exists a Lipschitz curve $\gamma \colon [0,1] \to X$ such that $\gamma(0) = x, \gamma(1) = y$ and $\operatorname{len}(\gamma) = d(x, y)$.

Any L-quasiconvex space is L-bi-Lipschitz to a geodesic space by defining a new distance

$$\overline{d}(x, y) = \inf_{\substack{\gamma \colon [0,1] \to X \\ \gamma(0)=x, \gamma(1)=y}} \operatorname{len}(\gamma).$$

Lemma 3.2.12. *If (X, d, μ) is a D-measure doubling metric measure space and f is a locally integrable function, and $B = B(x, r)$. Then for any $a \in \mathbb{R}$*

$$\frac{1}{2r} \fint_B |f - f_B| \, d\mu \leq \frac{1}{r} \fint_B |f - a| \, d\mu.$$

Proof: The result follows by twice applying the triangle inequality.

$$\frac{1}{2r} \fint_B |f - f_B| \, d\mu \leq \frac{1}{2r} \fint_B |f - a| + |a - f_B| \, d\mu \leq \frac{1}{r} \fint_B |f - a| \, d\mu$$

$\square$

3.3   Proving Poincaré Inequalities

We define a notion of connectivity in terms of an avoidance property. For the definition of curve fragments see Definition 3.2.6. Recall the main definition.

Definition 3.3.1. Let[5] $0 < \delta$, $0 < \epsilon < 1$ and $C > 1$ be given. If $(x, y) \in X \times X$ is a pair of points with $d(x, y) = r > 0$, then we say that the pair (x, y) is (C, δ, ϵ)-connected if for every Borel set E such that $\mu(E \cap B(x, Cr)) < \epsilon\mu(B(x, Cr))$ there exists a 1-Lipschitz[6] curve fragment $\gamma \colon K \to X$ connecting x and y, such that the following hold.

1. $\text{len}(\gamma) \le Cd(x, y)$

2. $|\text{Undef}(\gamma)| < \delta d(x, y)$

3. $\gamma^{-1}(E) \subset \{0, \max(K)\}$

We call (X, d, μ) a $(C, \delta, \epsilon, r_0)$-connected space, if every pair of points $(x, y) \in X$ with $r = d(x, y) \le r_0$ is (C, δ, ϵ)-connected.

We say that X is (uniformly) $(C, \delta, \epsilon, r_0)$-connected along S, if every $(x, y) \in X$ with $x \in S$ and with $d(x, y) \le r_0$ is (C, δ, ϵ)-connected in X.

If (X, d, μ) is $(C, \delta, \epsilon, r_0)$-connected for all r_0, we simply say that (X, d, μ) is (C, δ, ϵ)-connected.

Remark: Since we are working with Radon measures on proper metric spaces, to verify the condition we would only need to consider "test sets" E which are either all compact, or just open. For open sets this is trivial, since the measure is outer regular and Borel obstacle set E can be approximated on the outside by an open set E'. For compact sets, the argument goes via exhausting an open set by compact sets and obtaining a sequence of curve fragments.

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[5]The definition is only interesting for $\delta < 1$. However, we allow it to be larger to simplify some arguments below. If $\delta \ge 1$ then the curve fragment could consist of two points $\{0, d(x, y)\}$, and $\gamma(0) = x, \gamma(d(x, y)) = y$.

[6]According to Lemma 3.2.7 we could simply assume some Lipschitz bound, but it is easier to normalize the situation.

In some of our applications, one can work with Lipschitz curves instead of curve fragments and the reader is advised to first consider that case as it simplifies some arguments. For details, see Definition 1.2.35 and Theorem 3.3.4. In the case of curves we would replace $\gamma^{-1}(E) \subset \{0, \max(K)\}$ with $|\gamma^{-1}(E)| < \delta d(x, y)$. This would suffice for Theorem 3.3.17. However, this would not work for the relative notion of X uniformly connected along a subset S. This relative notion of connectivity arises in the context of differentiability spaces, see chapter 4. Also, it is easier to find curve fragments, because we are permitted to make small "jumps".

Our first observation is that the connectivity condition already implies a doubling bound.

Lemma 3.3.2. *Let (X, d, μ) be a complete metric space and (C, δ, ϵ)-connected for some $C \geq 2$ and $0 < \delta < 1$, then it is also D-measure doubling for some $D > 1$.*

Proof: Fix $(x, r) \in X \times (0, \infty)$. Define $r_0 = r$. We can assume without loss of generality that $B(x, r_0/2) \neq B(x, r_0)$. In other words there, is a $y \in B(x, r_0) \setminus B(x, r_0/2)$. Let $s = d(x, y)$. Define $E = B(x, \delta s)$. We will show that $\mu(B(x, \delta s)) \geq \epsilon B(x, r)$.

If $\mu(B(x, \delta s)) < \epsilon B(x, r) < \epsilon B(x, Cs)$, then there is a 1-Lipschitz curve fragment $\gamma \colon K \to X$ connecting x to y with $|\text{Undef}(\gamma)| < \delta s$, and with $\gamma^{-1}(E) \in \{0, \max(K)\}$. Now, $\gamma(0) = x$. We know that $(0, \delta s) \cap K \neq \emptyset$, because otherwise $\text{Undef}(\gamma)$ would be too big. Choose a $t \in (0, \delta s)$. But $d(\gamma(t), x) \leq t$ by the Lipschitz bound and $\gamma(t) \notin E$ by assumption. However, the Lipschitz bound gives also $\gamma(t) \in E$, which is a contradiction. Thus we obtain the lower bound for volume.

Next, define $r_1 = \delta s \leq \delta r_0$. Inductively we can define $r_k \leq \delta^k r_0$ such that $\mu(B(x, r_k)) \geq \epsilon B(x, r_{k-1})$. This gives, by setting $k = 1 - \frac{1}{\log_2(\delta)}$, that

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$$\mu(B(x, 1/2r)) \geq \mu(B(x, r_k)) \geq \epsilon^k \mu(B(x, r)).$$

This is the desired estimate 1.2.14.

$\square$

This lemma implies that the explicit doubling assumption in the theorems below is somewhat redundant. We leave it in order to explicate the dependence on the constants.

The definition of (C, δ, ϵ)-connectivity involves almost avoiding sets of a given relative size ϵ. One is led to ask if necessarily better avoidance properties hold for much smaller sets. This type of self-improvement closely resembles the improving properties of reverse Hölder inequalities and A_∞-weights [135, Chapter V]. It turns out that since the connectivity holds at every scale a maximal function argument can be used to improve the connectivity estimate and allows us to take δ arbitrarily small. We introduce the notion of finely α-connected metric measure space.

Definition 3.3.3. We call a metric measure space (X, d, μ) finely α-connected with parameters (C_1, C_2) if for any $0 < \tau < 1$ the space is $(C_1, C_2 \tau^\alpha, \tau)$-connected. Further, we say that the space is locally finely (α, r_0)-connected with parameters (C_1, C_2) if it is $(C_1, C_2 \tau^\alpha, \tau, r_0)$-connected for any $0 < \tau < 1$. If we do not explicate the dependence on the constants, we simply say (X, d, μ) is finely α-connected if constants $C_1, C_2, r_0 > 0$ exist with the previous properties. Similarly we define locally finely (α, r_0)-connected spaces.

Example: A direct computation shows that every n-dimensional manifold and every convex subset of $\mathbb{R}^n$ is finely $\frac{1}{n}$-connected. Clearly any finely (α, r_0)-connected

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space is also $(C, \delta, \epsilon, r_0)$-connected for some parameters but the converse is not obvious. We next establish this converse for doubling metric measure spaces. This is done by an argument of filling in gaps iteratively. This will require maintaining some estimates at smaller scales which uses a maximal function type argument. Since this argument will also be used later to prove the main Theorem 3.3.17, we will first explain it in the simple context of proving quasiconvexity. Quasiconvexity could also be proved by techniques of Sean Li and David Bate [12].

Theorem 3.3.4. *If (X, d, μ) is a proper locally finely (α, r_0)-connected metric measure space with parameters (C_1, C_2), then it is locally (L, r_0)-quasiconvex for $L = C_1$*

Proof: Take an arbitrary pair of points $(x, y) \in X \times X$ and denote $r = d(x, y) \le r_0$. Further, abbreviate $D_n = (C + \delta C + \delta^2 C + \cdots + \delta^n C)$ and $C_1 = \frac{C}{1-\delta}$. We define for each $n \ge 0$ inductively 1-Lipschitz curve fragments $\gamma_n \colon K_n \to X$ with the following properties.

1. $\min(K_n) = 0$, $\max(K_n) \le D_n d(x, y)$

2. $\mathrm{len}(\gamma_n) \le D_n d(x, y) \le C_1 d(x, y)$

3. $|\mathrm{Undef}(\gamma_n)| \le \delta^n d(x, y)$

Since (X, d) is proper, this suffices to prove the statement by taking a sub-sequential limit[7].

To initiate the iteration, define $K_0 = \{0, r\}, \gamma_0(0) = x, \gamma_0(r) = y$. Next, assume γ_n has been constructed, and denote $[0, \max(K_n)] = \bigcup_i (a_i, b_i)$, where (a_i, b_i) are

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[7]A more careful argument could remove taking a sub-sequence, and further would allow for finding curves defined on full measure subsets of an interval without completeness. However, these observations are a bit tedious and distract from the main argument.

disjoint, open and maximal intervals.

Dilation step: We will seek to patch each gap (a_i, b_i) with a new curve fragment but first need to stretch γ_n slightly. By Lemma 3.2.7 and Lemma 3.2.8 we can define a define a new curve fragment $\gamma'_n \colon K'_n \to X$ such that the following holds. Denote the maximal open intervals as $[0, \max(K'_n)] = \bigcup_i (a'_i, b'_i)$, where (a'_i, b'_i) corresponds to (a_i, b_i) via a re-parametrization. Then, we can ensure that $Cd(\gamma'_n(a'_i), \gamma'_n(b_i)) = |b'_i - a'_i|$ and further

$$\max(K'_n) \leq \max(K_n) + C\delta^n d(x, y) \leq D_{n+1} d(x, y) \tag{3.3.5}$$

Filling in: Next each of the gaps (a'_i, b'_i) is filled with a new curve fragment γ_i. Denote by $d_i = d(\gamma'(a'_i), \gamma'(b'_i)) = d(\gamma(a_i), \gamma(b_i)) \leq |b_i - a_i|$. By inductive hypothesis

$$\sum_i d_i \leq |\text{Undef}(\gamma_n)| \delta^n d(x, y) \tag{3.3.6}$$

By using (C, δ, ϵ)-connectivity with the obstacle $E = \emptyset$ and the pair of points $\gamma'(a'_i), \gamma'(b'_i)$ we find a 1-Lipschitz curve fragment $\gamma'_i \colon K_i \to X$ connecting $\gamma'(a'_i)$ to $\gamma'(b'_i)$ and $\text{len}(\gamma'_i) \leq Cd_i = |b'_i - a'_i|$ and $|\text{Undef}(\gamma'_i)| \leq \delta d_i$. By possibly dilating the curves a little more we can assume $\text{len}(\gamma'_i) = Cd_i = |b'_i - a'_i|$.

Next define $K_{n+1} = K'_n \bigcup (a'_i + K_i)$ and $\gamma_{n+1}(t) = \gamma'(t)$ for $t \in K'_n$ and $\gamma_{n+1}(t) = \gamma'_i(t - a'_i)$ for $t \in a'_i + K_i$. Then γ_{n+1} is easily seen to be a 1-Lipschitz curve fragment connecting x to y and defined on the compact set K_{n+1}. By Equation 3.3.5 above

$$\text{len}(\gamma_{n+1}) \leq \max(K'_n)$$
$$\leq D_{n+1}d(x,y).$$

Also we have
$$\text{Undef}(\gamma_{n+1}) = \bigcup_i \left(\text{Undef}(\gamma'_i) + a'_i\right),$$

and by Equation 3.3.6

$$|\text{Undef}(\gamma_{n+1})| \leq \sum_i |\text{Undef}(\gamma'_i)|$$
$$\leq \sum_i \delta d_i \leq \delta^{n+1}d(x,y)$$

This completes the recursive proof.

$\square$

The Proof of the Main Theorem 3.3.17 is essentially the same. However, we need to first prove an improved version of connectivity. Both arguments will involve the same scheme as above, but additional care is needed in showing that the curve fragments exist at smaller scales. The proofs are structured as inductive proofs instead of recursion.

Theorem 3.3.7. *Assume* $0 < \delta < 1, 0 < \epsilon, 0 < r_0, 1 < D$. *If* (X, d, μ) *is a* $(C, \delta, \epsilon, r_0)$-*connected* (D, r_0)-*doubling metric measure space, then there exist* $0 < \alpha, C_1, C_2$ *such that it is also finely* $(\alpha, r_0/(C_1 8))$-*connected with parame-*

ters (C_1, C_2). *We can choose* $\alpha = \frac{\ln(\delta)}{\ln(\frac{\epsilon}{2M})}$, $C_1 = \frac{C}{1-\delta}$ *and* $C_2 = \frac{2M}{\epsilon}$, *where*
$M = 2\max(D^{-\log_2(1-\delta)+1}, D^3)$.

Remark: We do not need it for the proof, but always $0 < \alpha \leq 1$. This could be seen by the argument in Lemma 3.3.2, which can be turned around to give a lower bound for δ in terms of ϵ and the optimal doubling constant.

Proof: Define C_1, C_2, M, α as above. Throughout the proof denote by E an arbitrary open set in Definition 3.1.2. We don't lose any generality by assuming that the obstacles which are tested are open. The proof is an iterative construction, where at every stage gaps in a curve are filled and the curve fragments are slightly stretched. This argument could be phrased recursively but we instead phrase it using induction. For the following statement abbreviate $D_n = (C + \delta C + \delta^2 C + \cdots + \delta^n C)$. Fix $M = 2\max(D^{-\log_2(1-\delta)+1}, D^3)$.

For any $n \in \mathbb{N}$ we will show the following induction statement $\mathcal{P}_n$: If $x, y \in X$, $d(x, y) = r < r_0/(C_1 8)$ and

$$\mu\left(E \cap B\left(x, C_1 r\right)\right) \leq \left(\frac{\epsilon}{2M}\right)^n \mu\left(B\left(x, C_1 r\right)\right),$$

then there is a 1-Lipschitz curve fragment $\gamma\colon K \to X$ connecting x to y such that the following estimate hold:

1. $\min(K) = 0, \max(K) \leq D_n d(x, y)$

2. $\operatorname{len}(\gamma) \leq D_n d(x, y) \leq C_1 d(x, y)$

3. $|\operatorname{Undef}(\gamma)| \leq \delta^n d(x, y)$

4. $\gamma^{-1}(E) \subset \{0, \max(K)\}$.

This clearly is sufficient to establish the claim. Also note that it results in the given value $\alpha = \frac{\ln(\delta)}{\ln\left(\frac{\epsilon}{2M}\right)}$.

Case $n = 1$: Note, $M > D^{-\log_2(1-\delta)+1}$. Then we get

$$
\begin{aligned}
\mu(E \cap B(x, Cr)) \;\leq\;& \mu(E \cap B(x, C_1 r)) \leq \frac{\epsilon}{2M} \mu(B(x, C_1 r)) \\
\leq\;& \frac{\epsilon}{2M} D^{-\log_2(1-\delta)+1} \mu(B(x, Cr)) \leq \frac{\epsilon}{2} \mu(B(x, Cr)).
\end{aligned}
$$

We have assumed that the space is (C, δ, ϵ)-connected and thus the desired 1-Lipschitz curve fragment exists.

Induction step, assume statement for n, and prove for $n + 1$: Take an arbitrary x, y, E with the property $d(x, y) = r < r_0$ and

$$
\mu\left(E \cap B\left(x, C_1 r\right)\right) \leq \left(\frac{\epsilon}{2M}\right)^{n+1} \mu\left(B\left(x, C_1 r\right)\right).
$$

Define the set

$$
E_M = \left\{ M(1_{E \cap B(x, C_1 r)}) \geq \left(\frac{\epsilon}{2M}\right)^{n} \right\} \cap B(x, C_1 r).
$$

By standard doubling and maximal function estimates in Lemma 3.2.9, if $M \geq D^3$, we have $\mu(E_M) \leq \epsilon/2\mu(B(x, C_1 r))$. Further by the case $n = 1$ there is a curve fragment $\gamma' \colon K' \to X$ connecting x to y with the properties

1. $\min(K') = 0, \max(K') \leq D_n d(x, y)$

2. $\text{len}(\gamma') \le Cd(x, y)$,

3. $|\text{Undef}(\gamma')| \le \delta d(x, y)$, and

4. $\gamma^{-1}(E_M) \subset \{0, \max(K)\}$.

The set $\text{Undef}(\gamma')$ is open and as such we can represent $\text{Undef}(\gamma') = \bigcup(a_i, b_i)$ with disjoint intervals such that $a_i, b_i \in K'$. We will seek to patch each gap (a_i, b_i) with a new curve fragment, but first need to stretch γ' slightly. This will proceed in the same two steps as before.

Dilation argument: Dilate the domain using Lemma 3.2.7 and Lemma 3.2.8. We will simplify notation by using the same symbols for the dilated curve. Then we can assume $d(\gamma'(a_i), \gamma'(b_i)) = D_n |b_i - a_i|$, $\min(K') = 0$ and

$$\max(K') \le Cd(x, y) + D_n \delta d(x, y) \le D_{n+1} d(x, y) \qquad (3.3.8)$$

Define $d(\gamma'(b_i), \gamma'(a_i)) = d_i$. By assumption on $\text{Undef}(\gamma')$ (before its dilation),

$$\sum_i d_i \le \delta d(x, y) \qquad (3.3.9)$$

Filling in: Further for each interval with $a_i \ne 0$ and $b_i \ne \max(K')$ both the endpoints $\gamma'(a_i), \gamma'(b_i)$ do not belong to E_M. If $a_i = 0$, then $\gamma'(b_i) \notin E_M$ and if $b_i = \max(K')$ then $\gamma'(a_i) \notin E_M$. This occurs only for possible terminal intervals. In any case, for one of the points $\gamma'(a_i), \gamma'(b_i) \notin E_M$. Say, $\gamma'(a_i) \notin E_M$. Then for the ball $B(\gamma'(a_i), Cd_i)$ we have

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$$\mu(E \cap B(\gamma'(a_i), Cd_i)) \le \left(\frac{\epsilon}{2E}\right)^n \mu(B(\gamma'(a_i), Cd_i)).$$

Thus by the induction hypothesis we can define a 1-Lipschitz curve fragments $\gamma'_i \colon K_i \to X$ connecting $\gamma'(a_i)$ to $\gamma'(b_i)$ such that

1. $\text{len}(\gamma'_i) \le D_n d_i$; and

2.
$$|[0, \max(K_i)] \setminus K_i| \le \delta^n d_i.$$

3. $\gamma'^{-1}_i(E) \subset \{a_i, b_i\}$

Similarly if $\gamma'(b_i) \notin E_M$. By possibly dilating the curves a little, while keeping the gaps the same size, we can assume $\text{len}(\gamma'_i) = D_n d_i$. This can be done by an argument similar to Lemma 3.2.8.

Next, define $K = K' \bigcup (a_i + K_i)$ and $\gamma(x) = \gamma'(x)$ for $x \in K'$ and $\gamma(x) = \gamma'_i(x - a_i)$ for $x \in a_i + K_i$. Then γ is easily seen to be a 1-Lipschitz curve fragment connecting x to y and defined on the compact set K. By Equation 3.3.8 above

$$\begin{aligned}
\text{len}(\gamma) & \le & \max(K') = \max(K) \\
& \le & D_{n+1} d(x, y).
\end{aligned}$$

Also we have
$$\text{Undef}(\gamma) = \bigcup_i \left(\text{Undef}(\gamma'_i) + a_i\right),$$

and by Equation 3.3.9

$$\begin{aligned}|\text{Undef}(\gamma)| \;&\leq\; \sum_i |\text{Undef}(\gamma'_i)| \\ &\leq\; \sum_i \delta^n d_i \leq \delta^{n+1} d(x,y)\end{aligned}$$

This completes the induction step.

$\square$

The main application of the previous theorems is the establishment of curves, or curve fragments, that avoid a set on which we have poor control of the oscillation of the function. We will present an example of this type of argument here.

Example: Fix $M > 1$. Let (X, d, μ) be finely (α, r_0)-connected with parameters (C_1, C_2). If $x, y \in X$,
$$\left(\fint_{B(x,C_1 r)} g^p \, d\mu \right)^{\frac{1}{p}} < \epsilon,$$
and $d(x,y) = r < r_0$, then there exists a curve fragment $\gamma\colon K \to X$ connecting x to y, such that

$$\int_\gamma g \, ds \leq C_1 M \epsilon d(x,y)$$

and $|\text{Undef}(\gamma)| < \frac{C_2}{M^{p\alpha}}$. Observe the emergence of p in the size of the gaps in K. For p larger the gaps become smaller.

Proof: Apply the definition of fine (α, r_0)-connectedness to the set $E = \{g > M\epsilon\}$,

which has measure at most $\frac{1}{M^p}\mu(B(x, C_1 r))$. Since $g \le M\epsilon$ on the resulting curve fragment γ and $\text{len}(\gamma) \le C_1 d(x, y)$ we obtain the desired integral estimate.

$\square$

As an immediate corollary we obtain the following lemma.

Lemma 3.3.10. *Assume (X, d, μ) is $(C, \delta, \epsilon, r_0)$-connected and D-measure doubling. Then there is a constant $M > 0$ such that it is also $(M, \delta, \bar{\epsilon}, r_0/M)$-connected by curves for some $M, \bar{\epsilon} >$. Conversely, if a space is $(C, \delta, \epsilon, r_0)$-connected by curves it is also $(C, \delta, \epsilon, r_0)$-connected.*

Proof: The second claim is easier to show. Given an obstacle E and the curve γ from Definition 3.1.3, we can restrict γ to a compact subset in $\gamma^{-1}(E)$ with almost full measure. Thus, we need to only prove the first claim.

By Theorem 1.2.38 we have that X is $(C_1, C_2\tau^\alpha, \tau, r_0/(8C_1))$-connected for every $\tau > 0$ and some constants (C_1, C_2). By Theorem 3.3.4 we have that X is locally $(L, r_0/(8C_1))$-quasiconvex for the same parameter C_1. Choose $\bar{\epsilon} = (\delta/(LC_2))^{\frac{1}{\alpha}}$. We now show that X is $(LC_1, \delta, \bar{\epsilon}, r_0/(8C_1))$-connected by curves. Let $x, y \in X$ be such that $d(x, y) = r \le r_0/(8C_1)$, and let $E \subset B(x, C_1 r)$ be such that $\mu(E) \le \bar{\epsilon}\mu(B(x, Cr))$. Then by the improved connectivity, we have a 1-Lipschitz curve fragment $\gamma \colon K \to X$ connecting x to y with

- $\text{len}(\gamma) \le C_1 r$

- $|\text{Undef}(\gamma)| \le C_1 \bar{\epsilon}^\alpha r \le \delta/Lr$

- $\gamma^{-1}(E) \subset \{\min(K), \max(K)\}$.

Choose maximal open and disjoint intervals as in Equation 1.2.31 $\operatorname{Undef}(\gamma) \subset$ $\bigcup_i (a_i, b_i)$. By a dilation of the domain we can assume $Ld(\gamma(a_i), \gamma(b_i)) = |b_i - a_i|$. Denote the dilated curve by the same symbol. We have after the dilation that $|\operatorname{Undef}(\gamma)| = \sum_i |b_i - a_i| \leq \delta/Ld(x,y) \cdot L \leq \delta d(x,y)$. By local L-quasiconvexity (Theorem 3.3.4) we can connect $\gamma(a_i), \gamma(b_i)$ by a 1-Lipschitz curve $\overline{\gamma}_i \colon [a_i, b_i] \to X$ of length at most $|b_i - a_i|$. Finally define $\overline{\gamma}(t) = t$ on K, and $\overline{\gamma}(t) = \overline{\gamma}_i(t)$ for $t \in [a_i, b_i]$. It is not hard to show that $\overline{\gamma}$ is 1-Lipschitz. Further

$$\operatorname{len}(\overline{\gamma}) \leq \operatorname{len}(\gamma) + \sum_i \operatorname{len}(\overline{\gamma}_i) \leq (1 + L)C_1 d(x,y),$$

and

$$|\gamma^{-1}(E)| \leq \sum_{i=1}^{n} |b_i - a_i| \leq \delta d(x,y).$$

This is all that is needed to verify the conditions in Definition 1.2.35.

$\square$

Theorem 3.3.11. *Assume (X, d, μ) is locally finely (α, r_0)-connected with parameters (C_1, C_2) and D-measure doubling. For any $p > \frac{1}{\alpha}$ there exists a constant C_3 with the following property. If $x, y \in X$ are points with $d(x,y) = r < r_0/(8C_1)$, and g is a positive Borel function with*

$$\left(\fint_{B(x, 2C_1 r)} g^p \, d\mu \right)^{\frac{1}{p}} \leq 1,$$

then there exists a 1-Lipschitz curve $\gamma \colon [0, L] \to X$ connecting x to y, such that

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$$\int_\gamma g \ ds \le C_3 d(x, y).$$

We can set $C_3 = \frac{C_1 M}{1 - M\delta}$, where $M = 2(D^4)^{\frac{1}{p\alpha - 1}}$ and $\delta = \left(\frac{D^4}{M^p}\right)^\alpha$.

Remark: By possibly scaling g we can use the statement for non-unit L^p-bounds for g.

Proof: One can approximate g from above by a lower-semi-continuous function, and then from below by an increasing sequence of continuous functions. This standard argument (see e.g. [57, Chapter 1], [74, Proposition 2.27]) allows for assuming that g is continuous. Fix $M = 2(D^4)^{\frac{1}{p\alpha - 1}}$ and choose a $\delta = \left(\frac{D^4}{M^p}\right)^\alpha$. We have $\frac{1}{M} > \delta > 0$, and $M \ge 2$.

We will construct the curve by an iterative procedure depending on n. The proof is structured by an inductive statement. The procedure is the same as in the previous two proofs: a dilation argument followed by gap-filling. However, a new issue arises as we need to ensure that the integral estimate holds at the smaller scale. Abbreviate

$$D_n = (C_1 M + C_1 M (M\delta) + \cdots + C_1 M (M\delta)^{n-1}),$$

$$C_n = (C_1 + C_1(\delta) + \cdots + C_1(\delta)^{n-1})$$

and

$$D_\infty = \lim_{n \to \infty} D_n, C_\infty = \lim_{n \to \infty} C_n.$$

Induction statement: $\mathcal{P}_n$: Assume $v, w \in X$ are arbitrary with $d(v, w) = r < r_0/(8C_1)$ and h is any continuous function on X with

$$\left(\fint_{B(v, 2C_1 r)} h^p \, d\mu \right)^{\frac{1}{p}} \leq 1,$$

. Then, there is a 1-Lipschitz curve fragment $\gamma \colon K \to X$ with $\min(K) = 0, \max(K) \leq C_n d(x, y)$,

$$\int_\gamma h \, ds \leq D_n d(v, w), \tag{3.3.12}$$

$$|\mathrm{Undef}(\gamma)| \leq \delta^n d(v, w), \tag{3.3.13}$$

$$\mathrm{len}(\gamma) < C_n d(v, w) \tag{3.3.14}$$

and for any maximal open set $(a_i, b_i) \subset \mathrm{Undef}\gamma$ we have $d(\gamma(a_i), \gamma(b_i)) = |b_i - a_i| = d_i$ and for at least one $z_i \in \{a_i, b_i\}$

$$\left(\fint_{B(z_i, 2C_1 d_i)} h^p \, d\mu \right)^{\frac{1}{p}} \leq M^n. \tag{3.3.15}$$

Once we have shown $\mathcal{P}_n$ for every n we can apply it with $(x, y) = (v, w)$ and $h = g$ to obtain a sequence of curve fragments γ_n with $\int_{\gamma_n} g \, ds \leq D_n d(x, y)$, and by taking a limit we obtain a curve fragment γ with $\mathrm{len}(\gamma) \leq C_\infty d(x, y)$ (from 3.3.14) and $\int_\gamma g \, ds \leq D_\infty d(x, y)$ (from 3.3.12). Finally, from 3.3.13 we can conclude that

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$|\mathrm{Undef}(\gamma)| = 0$, and thus γ is in fact a curve. In the rest of the proof we show by induction $\mathcal{P}_n$ for every natural number n.

Base case $n = 1$: Define $E_M = \{M_{2C_1\delta r}h^p > M^p\}$. Then

$$\mu(E_M \cap B(v, C_1 r)) \leq \frac{D^3}{M^p}\mu(B(v, 2C_1 r)) \leq \frac{D^4}{M^p}\mu(B(v, C_1 r))$$

by Theorem 3.2.9. Thus, by the fine connectivity, there is a $1-$Lipschitz curve fragment connecting v to w with

$$\mathrm{len}(\gamma) \leq C_1 d(v, w),$$

$$|\mathrm{Undef}(\gamma)| < \frac{(D^4)^\alpha}{M^{\alpha p}},$$

and

$$\gamma(K \setminus \{0, \max(K)\}) \subset \{M_{2C_1\delta r}h \leq M\}.$$

This immediately established 3.3.14. Next, we prove Estimates 3.3.13 and 3.3.12.

The estimate 3.3.13 follows from the choice $\delta = \left(\frac{D^4}{M^p}\right)^\alpha$. By assumption, $\gamma(K \setminus \{0, \max(K)\}) \subset \{M_{2C_1 r}h^p \leq M^p\}$ and h is continuous. Thus, it follows that $h \circ \gamma \leq M$ on $K \setminus \{0, \max(K)\}$. In particular, we obtain 3.3.12 by the simple upper bound

$$\int_{\gamma} h \ ds \leq \mathrm{len}(\gamma)M \leq C_1 M d(x, y).$$

Finally, we show Estimate 3.3.15. Let $(a_i, b_i) \subset [0, \max(K)] \setminus K$ be an arbitrary maximal open interval. By using Lemma 3.2.7 we can assume $d(\gamma(a_i), \gamma(b_i)) = |b_i - a_i| = d_i$. Also, either one of a_i, b_i is not in $\gamma^{-1}(E_M)$. Denote it by z_i. The maximal function estimate $M_{2C_1 \delta r}(z_i) \leq M^p$ combined with $d_i \leq \delta r$ gives

$$\left(\fint_{B(z_i, 2C_1 d_i)} h^p \ d\mu \right)^{\frac{1}{p}} \leq M^p.$$

Assume $\mathcal{P}_n$ and show $\mathcal{P}_{n+1}$: By the case $\mathcal{P}_n$ there exists a 1-Lipschitz curve fragment $\gamma' \colon K \to X$ connecting v to w such that

$$\int_{\gamma'} h \ ds \leq D_n d(v, w),$$

$$\mathrm{len}(\gamma') \leq C_n d(v, w),$$

$$|\mathrm{Undef}(\gamma')| \leq \delta^n d(v, w),$$

and for any maximal open set $(a_i, b_i) \subset \mathrm{Undef}\gamma$ we have $d(\gamma(a_i), \gamma(b_i)) = |b_i - a_i| = d_i$ and at least for one $z_i \in \{a_i, b_i\}$

$$\left(\fint_{B(z_i, 2C_1 d_i)} h^p \ d\mu \right)^{\frac{1}{p}} \leq M^n.$$

Filling in gaps: Now similar to the argument in Lemma 1.2.40 we will fill in the gaps of γ'. The set $\mathrm{Undef}(\gamma')$ is open, and we can represent $\mathrm{Undef}(\gamma') = \bigcup (a_i, b_i)$

with disjoint intervals such that $a_i, b_i \in K'$. Define $|b_i - a_i| = d_i$. By 3.3.15 for one $z_i \in \{a_i, b_i\}$ we have

$$\left(\fint_{B(\gamma(z_i), 2C_1 d_i)} h^p \, d\mu \right)^{\frac{1}{p}} \leq M^n.$$

By the base case $n = 1$ (applied to the re-scaled g/M^n and $v = \gamma(a_i), w = \gamma(b_i)$) we can define 1-Lipschitz curve fragments $\gamma_i' \colon K_i \to X$ connecting $\gamma'(a_i)$ to $\gamma'(b_i)$ such that

- $\int_{\gamma_i'} g \, ds \leq C_1 M^n d_i$, and

-
$$|[0, \max(K_i)] \setminus K_i| \leq \delta d_i.$$

- For any maximal open sets $(a_j^i, b_j^i) \subset \mathrm{Undef} \gamma_i'$ we have $d(\gamma(a_j^i), \gamma(b_j^i)) = |a_j^i - a_j^i| = d_j^i$ and at least for one $z_j^i \in \{a_j^i, b_j^i\}$

$$M_{2C_1 d_j^i} h^p(z_j) \leq M^{(n+1)p}.$$

Adjustment: First, choose a parametrization by using Lemmas 3.2.7 and 3.2.8 such that $d(\gamma'(a_i), \gamma'(b_i)) = \mathrm{len}(\gamma_i')$. Next define $K = K' \bigcup (a_i + K_i)$, and define a curve fragment $\gamma \colon K \to X$ by $\gamma(t) = \gamma'(t)$ for $t \in K$ and $\gamma(t) = \gamma_i'(t - a_i)$ for $t \in a_i + K_i$. Then γ is easily seen to be a 1-Lipschitz curve fragment connecting v to w. The domain K is also clearly compact. We prove next the Inequality 3.3.12 for $n + 1$. Note that by the argument of Lemma 3.2.8 and continuity h combined with $\gamma_i'(K_i) \subset \{h \leq M^{n+1}\}$ we get

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$$\int_{\gamma_i} h\ ds \leq \mathrm{len}(\gamma_i)M^{n+1} \leq C_1\delta d_i M^{n+1}d(v,w).$$

Thus adding up the individual contributions and noting that $\sum_i d_i \leq \delta^n d(v,w)$ by assumption we obtain the desired estimate.

$$
\begin{aligned}
\int_\gamma h\ ds &= \int_{\gamma'} h\ ds + \sum_i \int_{\gamma_i} h\ ds \\
&\leq D_n d(v,w) + C_1\delta^n M^{n+1}d(v,w) \\
&\leq (C_1 M + C_1 M(M\delta) + \cdots + C_1 M(M\delta)^{n+1})d(v,w). \\
&= D_{n+1}d(v,w)
\end{aligned}
$$

Now prove the estimate for gaps 3.3.13 for $n+1$. Note that

$$\mathrm{Undef}(\gamma) = \bigcup_i \left(\mathrm{Undef}(\gamma_i') + \phi(a_i)\right),$$

and as such we obtain the desired estimate

$$
\begin{aligned}
|\mathrm{Undef}(\gamma)| &\leq \sum_i |\mathrm{Undef}(\gamma_i')| \\
&\leq \sum_i \delta^n|b_i - a_i| \leq \delta^{n+1}d(v,w)
\end{aligned}
$$

Finally, check the Estimate 3.3.15 concerning the integral average. Any maximal open interval $I \subset \mathrm{Undef}\,\gamma$ is also a translate of a maximal undefined interval for γ_i'. I.e. there is of the form $I = (a_j^i, b_j^i)$ for some maximal open interval $(a_j^i, b_j^i) \subset \mathrm{Undef}(\gamma_i') + a_i$. The desired Estimate 3.3.15 is equivalent to the desired

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estimate for γ_i'.

$\square$

Lemma 3.3.16. *Let (X, d, μ) be a metric measure space. If f is a Lipschitz function, and $\gamma \colon K \to X$ is a 1-Lipschitz curve fragment, then for every $a, b \in K$*

$$|f(\gamma(a)) - f(\gamma(b))| \leq \mathrm{LIP}\ f|\mathrm{Undef}(\gamma) \cap [a, b]| + \int_\gamma \mathrm{Lip}\ f\ ds.$$

Proof: Obvious by an integration argument.

$\square$

Finally, we prove the following quantitative version of Theorem 1.2.38, from which it follows immediately.

Theorem 3.3.17. *(Main Theorem) Let (X, d, μ) be a D-measure doubling locally $(C, \delta, \epsilon, r_0)$-connected metric measure space, then there exists C_{PI}, C_1 for $\alpha = \log_{\frac{\epsilon}{2D}}(\delta)$, $p > \frac{1}{\alpha}$ and $r < \frac{r_0}{8C_1}$, such that for any Lipschitz function f and any $x \in X$*

$$\fint_{B(x,r)} |f - f_{B(x,r)}|d\mu \leq C_{PI} r \left(\fint_{B(x,2C_1 r)} \mathrm{Lip}\ f^p \right)^{\frac{1}{p}}.$$

We can set $C_1 = \frac{C}{1-\delta}$ and $C_{PI} = 2C_3$ where C_3 is given by Lemma 3.3.11. In particular, the space is a PI-space.

Proof: Denote

$$A = \left(\fint_{B(x,2C_1r)} \text{Lip } f^p \right)^{\frac{1}{p}}.$$

Let $y \in B(x,r)$ be arbitrary. First use Lemma 3.3.11 on $h = \text{Lip } f/A$ to give a curve γ connecting x to y with

$$\int_\gamma \text{Lip } f \ ds \leq C_3 A r.$$

Thus, using Lemma 3.3.16 we get a segment inequality in terms of a maximal function

$$\fint_{B(x,r)} |f(y) - f(x)| \ d\mu \leq C_3 r A.$$

Together with Lemma 3.2.12 and the choice $a = f(x)$ this completes the proof.

$\square$

Similar techniques give a modulus estimate. For any $x, y \in X$, and any C, define the collection of curves.

$$\Gamma_{x,y,C} = \{\gamma \colon [0,1] \to X | \gamma(0) = x, \gamma(1) = y, \text{len}(\gamma) \leq Cd(x,y)\}$$

We call a non-negative measurable function ρ admissible for the family $\Gamma_{x,y,C}$ is for any $\gamma \in \Gamma_{x,y,C}$ we have

$$\int_\gamma \rho \ ds \geq 1.$$

The p-modulus of the curve family (centered at x, and scale Cr) is then defined as

$$\mathrm{Mod}_p^{x,s}(\Gamma_{x,y,C}) = \inf_{\rho \text{ is admissible for } \Gamma_{x,y,C}} \fint_{B(x,s)} \rho^p \, d\mu.$$

By the previous arguments for a finely α-connected space we can show the following lower bound for $p > \frac{1}{\alpha}$.

Theorem 3.1.6. *If (X, d, μ) is a complete (D, r_0)-measure doubling metric mea- sure space which is also locally finely (α, r_0)-connected with parameters (C_1, C_2), then for any $x, y \in X$ with $d(x, y) = r < r_0/(8C_1)$, any $p > \frac{1}{\alpha}$ we have*

$$\mathrm{Mod}_p^{x,2C_1 r}(\Gamma_{x,y,C_3}) \geq \frac{C_3^p}{r^{p-1}},$$

where C_3 is as in Lemma 3.3.11.

Proof: Immediate corollary of Lemma 3.3.11. If ρ is admissible, and

$$\fint_{B(x,2C_1 r)} \rho^p \, d\mu \leq \frac{1}{2C_3^p r^p},$$

then there exists a 1-Lipschitz curve γ connecting x to y of length at most $C_3 r$ and

$$\int_\gamma \rho \, ds < \frac{1}{C_3 r} C_3 r \leq 1,$$

which contradicts the admissibility of ρ. Thus, the original modulus estimate must hold.

$\square$

Remark: The results of this section are tight in terms of the range of p. We give two examples. Taking two copies of $\mathbb{R}^n$ glued along the origin (in fact any subset)

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we obtain a space that is $\frac{1}{n}$-connected and a simple analysis based on modulus estimates tells us that it only admits a $(1, p)$-Poincaré inequality for $p > n$. Another example arises from A_s weights w on $\mathbb{R}$. We know that such weights are $\frac{1}{s}$-connected [135], and thus they admit a Poincaré inequality for $p > s$. These examples suggest that the fine α-connectivity measures the stable regime of Poincaré inequalities, i.e. those p such that $(1, p)$-inequalities hold on glued spaces or spaces with controlled density changes.

For purposes of the following theorems we define an upper gradient of a Lipschitz function [74].

Definition 3.3.18. Let (X, d, μ) be a metric measure space and $C \geq 1$ some constant. Let $\Phi, \Psi \colon [0, \infty) \to [0, \infty)$ be increasing functions with the following properties.

- $\lim_{t \to 0} \Phi(t) = \Phi(0) = 0$

- $\lim_{t \to 0} \Psi(t) = \Psi(0) = 0$

We say that (X, d, μ) satisfies a non-homogeneous (Φ, Ψ, C)-Poincaré-inequality if for every 2-Lipschitz[8] function $f \colon X \to \mathbb{R}$, every upper gradient $g \colon X \to \mathbb{R}$ and every ball $B(x, r) \subset X$ we have

$$\fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq r \Psi \left(\fint_{B(x,Cr)} \Phi \circ g \, d\mu \right).$$

We get the following theorems.

---

[8]The constant 2 is only used to simplify arguments below. Any fixed bound could be used. Also, by adjusting the right-hand side by scaling with the Lipschitz constant of f, the inequality could be made homogeneous.

Theorem 1.2.44. *Suppose that (X, d, μ) is a quasiconvex D-doubling metric measure space and satisfies a non-homogeneous (Φ, Ψ, C)-Poincaré-inequality. Then (X, d, μ) is (C, δ, ϵ)-connected and moreover admits a true $(1, q)$-Poincareé inequality for some $q > 1$. All the variables are quantitative in the parameters. In particular, a non-homogeneous Poincaré-inequality in the sense of [12] implies a true Poincaré-inequality.*

Proof of Theorem 1.2.44: By a bi-Lipschitz change of parameters, and adjusting Ψ, Φ, we can assume that the space is geodesic. Further, by adding an increasing positive function to Ψ, Φ we can assume that $\Psi(t) > 0, \Phi(t) > 0$ for $t > 0$. By weakening the assumption, we can take Ψ, Φ to be upper semi-continuous. Assume next that for any 1-Lipschitz f and g an upper gradient and any ball $B(x, r)$.

$$\fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq r\Psi \left(\fint_{B(x,Cr)} \Phi(g) \, d\mu \right).$$

Define the right inverses $\xi(t) = \inf_{\Psi(s) \geq t} s$. Then we have $\xi(\Psi(t)) \leq t$ and thus

$$\xi \left(\frac{1}{r} \fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \right) \leq \fint_{B(x,r)} \Phi(g) \, d\mu.$$

Since $\lim_{t \to 0} \Phi(t) = 0$ we can choose a function $\sigma(t)$ such that for any $0 < s \leq \sigma(t)$ we have

$$\Phi(s) \leq t.$$

We will show $(B, \frac{1}{2}, \epsilon)$-connectivity for $0 < \epsilon$ small enough for a specific B. Our choice will be

$$B = \max \left(C, \left[\sigma \left(\xi \left(\frac{1}{20D^5} \right) / 2 \right) \right]^{-1} \right).$$

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Suppose X is not $(B, \frac{1}{2}, \epsilon)$-connected. If this were the case, we could choose $x, y \in X$ with $r = d(x, y)$, and an obstacle E such that

$$\frac{\mu(E \cap B(x, Br))}{\mu(B(x, Br))} \leq \epsilon,$$

and such that for any 1-Lipschitz curve fragment $\gamma \colon [0, L] \to B(x, Br)$ connecting x to y with $\operatorname{len}(\gamma) \leq Br$ we'd have $|\gamma^{-1}(E)| > \frac{r}{2}$. Define

$$\rho(z) = \inf_{\gamma \colon [0, L] \to X} \frac{1}{2B} \operatorname{len}(\gamma) + |\gamma^{-1}|(E).$$

Here, $\gamma \colon [0, L] \to X$ is a 1-Lipschitz curve such that $\gamma(0) = x, \gamma(L) = z$. It is easy to see that ρ is 2-Lipschitz, and has upper gradient $g = 1_E + \frac{1}{B}$. Further $\rho(0) = 0$. By assumption, there is no path γ connecting x to y with $|\gamma| \leq Br$ and $|\gamma^{-1}|(E) \leq \frac{r}{2}$, thus either of these inequalities fails, and we have

$$\rho(y) \geq \frac{r}{2}.$$

By the Lipschitz-condition, we have

$$\rho(z) \leq \frac{r}{10},$$

for $z \in B(x, \frac{r}{20})$ and

$$\rho(z) \geq \frac{2r}{5}$$

for $z \in B(y, \frac{r}{20})$. Thus, by a doubling estimate $\frac{1}{2r} \fint_{B(x, 2r)} |f - f_{B(x, 2)}| d\mu \geq \frac{1}{20D^5}$. On the other hand

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$$\fint_{B(x,2Cr)} \Phi(g)\, d\mu \;\leq\; \Phi\left(\frac{1}{B}\right) + \Phi(1)\frac{\mu(E)}{\mu(B(x,Cr))}$$

$$\leq\; \Phi\left(\sigma\left(\xi\left(\frac{1}{20D^5}\right)/2\right)\right) + D^{\log_2(B/C)+1}\Phi(1)\epsilon.$$

$$\leq\; \xi\left(\frac{1}{20D^5}\right)/2 + D^{\log_2(B/C)+1}\Phi(1)\epsilon.$$

Combining we would get

$$\xi\left(\frac{1}{20D^5}\right) \leq \xi\left(\frac{1}{20D^5}\right)/2 + D^{\log_2(B/C)+1}\Phi(1)\epsilon.$$

If $\epsilon < \frac{\xi\left(\frac{1}{20D^5}\right)}{2\Phi(1)D^{\log_2(B/C)+1}}$, this inequality fails deriving a contradiction and thus proving the conclusion.

$\square$

Theorem 1.2.38. *A (D, r_0)-doubling complete metric measure space (X, d, μ) admits a local $(1, p)$-Poincaré-inequality for some $1 \leq p < \infty$ if and only if it is locally (C, δ, ϵ)-connected for some $0 < \delta, \epsilon < 1$. Both directions of the theorem are quantitative in the respective parameters.*

Proof: The converse statement was proved in 3.3.17. Thus, we only prove that a measure-doubling space admitting a $(1, p)$ Poincaré-inequality for some $1 \leq p < \infty$ also admits (C, δ, ϵ)-connectivity for some constants. This is a particular case of the more general Theorem 1.2.44 by setting $\Phi(x) = x^p$ and $\Psi(x) = C_{PI}x^{\frac{1}{p}}$ here C_{PI} is the constant in the Poincaré inequality. The assumption of geodesic

is attained by a bi-Lipschitz deformation. However, there is an issue related to whether the Poincaré inequality is assumed for Lipschitz functions or all continuous functions and their upport gradients. We use the latter for non-homogeneous Poincaré inequalities. In the case of a complete metric space these are equivalent by [84, Theorem 2].

$\square$

3.4 Corollaries and Applications

If X is a complete and proper metric space, we denote by $\Gamma(X)$ the space of geodesics parametrized by the interval $[0, 1]$. This space can be made into a complete proper metric space by using the distance $d(\gamma, \gamma') = \sup_{t \in [0,1]} d(\gamma(t), \gamma'(t))$. Further define for each $t \in [0, 1]$ the map $e_t \colon \Gamma(X) \to X$ by $e_t(\gamma) = \gamma(t)$.

Definition 3.4.1. Let ρ be a decreasing function. A proper geodesic metric measure space (X, d, μ) is called a measure $\rho(t)$-contraction space, if for every $x \in X$ and every $B(y, r) \subset X$, there is a Borel probability measure Π on $\Gamma(X)$ such that $e_0(\gamma) = x$, Π-almost surely, and $e_1^* \Pi = \frac{\mu|_{B(y,r)}}{\mu(B(y,r))}$, and for any $t \in (0, 1)$ we have

$$e_t^* \Pi \leq \frac{\rho(t)}{\mu(B(y, r))} d\mu.$$

This class of metric measure spaces includes the MCP-spaces by Ohta [113], as well as $CD(K, N)$ spaces, and any Ricci-limit, or Ricci-bounded space. In fact this class of spaces is more general than the MCP-spaces considered by Ohta. Until now it was unknown, if such spaces possess Poincaré inequalities. We show as

a corollary of Theorems 1.2.38 and 1.2.41 that this indeed holds. The definition could be modified to permit a probability measure on C-quasiconvex paths, but we don't need this result here.

Corollary 3.4.2. *Any measure $\rho(t)$-contraction space (X, d, μ) admits a $(1, p)$-Poincaré inequality.*

Proof: Clearly any such space is $\rho(1/2)$-doubling. We will show that the space is $\left(2, \delta, \frac{\delta}{100\rho(1/2)^6\rho(\delta/3)}\right)$-connected. Consider any $x, y \in X$, and $d(x, y) = r$, and a set $E \subset B(x, 2r)$ with $\mu(E) \leq \frac{\delta}{100\rho(1/2)^6\rho(\delta/3)}\mu(B(x, 2r))$. It is sufficient to find a 1-Lipschitz curve $\gamma \colon [0, L] \to X$ connecting x to y with $|\gamma^{-1}(E)| < \delta d(x, y)$, since then a curve fragment can be defined by restricting γ to an appropriate compact set $K \subset [0, L] \setminus E$.

Consider a midpoint z on a geodesic connecting x to y and the ball $B(z, r/2)$. Apply the definition of ρ-contraction space to a point x and the ball $B(z, r/2)$. This constructs a probability measure Π on curves from x to $B(z, r/2)$, such that

$$e_t^*\Pi \leq \frac{\rho(t)}{\mu(B(z, r/2))}d\mu.$$

Note that $\gamma(\delta/3) \in B(x, \delta/3r)$ Π-almost surely. Thus

$$\int \left|[\delta/3, 1] \cap \gamma^{-1}(E)\right| d\Pi_\gamma \ = \ \int_{\delta/3}^1 1_E(\gamma(t))dt d\Pi_\gamma \leq \frac{\rho(\delta/3)2}{\mu(B(z, \frac{r}{2}))}\mu(E)$$
$$\leq \ \frac{\delta}{50}.$$

Thus with probability strictly bigger than $\frac{1}{2}$ we have $|[\delta/3, 1] \cap \gamma^{-1}(E)| \leq \frac{\delta}{10}$. Let S_x be the set of end points in $B(z, r/2)$ with such a curve to x. Similarly

construct S_y can be constructed. By the volume estimates $S_x \cap S_y \neq \emptyset$. We can thus find a common point $w \in B(z, r/2)$, and curve fragments from $B(x, \delta/3)$ to w and from w to $B(y, \delta/3)$. This curve fragment once dilated by r results in the desired 1-Lipschitz curve fragment.

$\square$

Corollary 3.4.3. *If $\rho(t) = t^{-n}$, and (X, d, μ) is a ρ-contraction space, then it satisfies a Poincaré inequality for $p > \frac{1}{n+1}$. Since $MCP(0, n)$-spaces satisfy this assumption we conclude that they admit such a Poincaré inequality. The same holds for every $MCP(K, n)$-space, but with a local Poincaré inequality.*

Proof: A slightly more detailed version of the previous proof verifies the desired connectivity estimate, and gives that the space is finely α-connected for $\alpha = \frac{1}{n+1}$. The result follows from Theorem 1.2.41. For $K > 0$ the latter result is obvious and $K < 0$ we modify it slightly to obtain that the space is locally finely (α, r_0)-connected (see [113] for the modified ρ).

$\square$

Remark: It would seem that this result should be true for $p > n$, and probably $p \geq 1$, but methods do not yield these sharper results.

We next prove a result on the existence of Poincaré inequalities on spaces deformed by Muckenhoupt-weights.

Definition 3.4.4. Let (X, d, μ) be a D-measure doubling metric measure space. We say that a Radon measure ν is a generalized $A_\infty(\mu)$-measure, or $\nu \in A_\infty(\mu)$,

if $\nu = w\mu$, and there exist $0 < \epsilon < 1, 0 < \delta < 1$ such that for any $B(x,r)$ and any Borel-set $E \subset B(x,r)$

$$\nu(E) \le \delta\nu(B(x,r)) \Longrightarrow \mu(E) \le \epsilon\mu(B(x,r)).$$

In a geodesic doubling space any generalized $A_\infty(\mu)$-measure is doubling. The assumptions can be slightly weakened but counter-examples can be constructed if (X, d, μ) is just doubling.

Lemma 3.4.5. *Let (X, d, μ) be a D-measure doubling geodesic metric measure space. Then any Radon measure $\nu \in A_\infty(\mu)$ is doubling.*

Proof: Let the parameters corresponding to ν be δ, ϵ. By [43, Lemma 3.3][9] for $\eta \le 2^{\frac{\log_2(1-\epsilon)}{\log_2(1+D^{-5})}-2}$.

$$\mu(B(x,r) \setminus B(x,(1-\eta)r)) < (1-\epsilon)\mu(B(x,r)).$$

Thus $\mu(B(x,(1-\eta)r)) > \epsilon\mu(B(x,r))$, and by the definition $\nu(B(x,(1-\eta)r)) > \delta\nu(B(x,r))$. Iterating this estimate $N = -1/\log_2(1-\eta)+1$ times, we get

$$\nu(B(x,r/2)) \ge \delta^N \nu(B(x,r)).$$

This gives the desired doubling property.

□

We need another technical Lemma from the paper of Kansanen and Korte [81].

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[9]Proof works for geodesic metric spaces. See also [28, Proposition 6.12].

This is a metric space generalization of the classical property of self-improvement for A_∞-weights in Euclidean space [135].

Lemma 3.4.6. *[81] Let (X, d, μ) be a D-measure doubling geodesic metric measure space. Then for any Radon measure $\nu \in A_\infty(\mu)$ and for any $0 < \tau < 1$ there exists a $0 < \delta < 1$, such that for any $B(x, s)$ and any $E \subset B(x, s)$*

$$\nu(E) \leq \delta\nu(B(x, s)) \implies \mu(E) \leq \tau\mu(B(x, r)).$$

Theorem 3.4.7. *If (X, d, μ) is a geodesic PI-space and if $\nu \in A_\infty(\mu)$, then (X, d, ν) is also a PI-space*

Proof: By Lemma 3.4.6 we see that (X, d, ν) is doubling, and from Theorem 1.2.38 we get that (X, d, μ) is $(C, \frac{1}{2}, \epsilon_\mu)$-connected for some C, ϵ_μ. Further by Lemma 3.4.6 we have a ϵ_ν such that for any Borel $E \subset B(x, r) \subset X$

$$\nu(E) \leq \epsilon_\nu\nu(B(x, r)) \implies \mu(E) \leq \epsilon_\mu\mu(B(x, r)).$$

Thus, since any obstacle E with volume density ϵ_ν with respect to ν will be an obstacle with volume density ϵ_μ with respect to μ, it is easy to verify that (X, d, ν) is $(C, \delta, \epsilon_\nu)$-connected. All the parameters can be made quantitative.

$\square$

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Chapter 4

Geometry of RNP-differentiability spaces and PI-rectifiability

4.1 Relationship between differentiability spaces and PI-spaces

Cheeger [28] defined a metric measure analog of a differentiable structure and proved a powerful generalization of Rademacher's theorem for PI-spaces. The spaces admitting differentiation are called *differentiability spaces* (or Lipschitz differentiability spaces [85]). See below at Definition 4.2.2 and 4.2.3. An early question was if the assumptions of Cheeger were in some sense necessary [72]. Because positive measure subsets of PI-spaces may be totally disconnected while they remain differentiability spaces [14], strictly speaking, a Poincaré inequality can not be necessary. However, it remained a question if a PI-space structure could be

recovered in a weaker form such as by taking tangents or by covering the space in some form. Various authors, such as Heinonen, Kleiner, Cheeger and Schioppa, posed similar questions. A strong form of this question appeared in [38], where it was asked if every differentiability space is PI-rectifiable. Call a metric space *PI-rectifiable* if it is rectifiable in terms of PI-spaces. Strong similarities between PI-spaces and Poincaré inequalities were discovered: asymptotic doubling [14], an asymptotic lip-Lip equality almost everywhere [11, 126], large family of curve fragments representing the measure and "lines" in the tangents [38]. This lent support for the rectifiability question.

Sean Li and David Bate tackled the question with the a priori stronger assumption of differentiability of Lipschitz maps with their range contained in a RNP-Banach space (*Radon Nikodym Property*) [12]. For more information on such Banach spaces see [116]. The question of differentiability of RNP-Banach space valued Lipschitz functions arose in the work of Kleiner and Cheeger where it was shown that a PI-space admits a Rademacher theorem for such functions [33]. Li and Bate managed to prove a partial converse by showing that the tangents of RNP-differentiability spaces admit a weak form of Poincaré inequality and that the spaces themselves admit a form of non-homogeneous Poincaré inequality infinitesimally. This already showed that differentiability was closely connected to Poincaré-type inequalities. However, it left open a question of Kleiner and Cheeger, if differentiability spaces were PI-rectifiable, i.e. if they can be covered by positive measure subsets of PI-spaces.

We improve both of these results by showing that RNP-differentiability spaces are in fact PI-rectifiable and almost everywhere their tangents are PI-spaces. The latter result is a consequence of the first.

Theorem 1.2.62. *A complete metric measure space (X, d, μ) is a RNP-Lipschitz differentiability space if and only if it is PI-rectifiable and every porous set has measure zero.*[1]

In other words, for RNP-differentiability spaces the conditions of Cheeger are necessary and sufficient up to decomposition and embedding. This will be a corollary of the more general classification result of PI-rectifiable spaces.

Theorem 1.2.61. *Let (X, d, μ) be an RNP-differentiability space. Then X can be covered by positive measure subsets V_i, such that each V_i is metric doubling, when equipped with its restricted distance, and for μ-a.e. $x \in V_i$ each space $M \in T_x(V_i)$ admits a PI-inequality of type $(1, p)$ for all p sufficiently large.* [2]

This statement also has a direct proof that is based on the asymptotic Poincaré inequality discovered by Bate and Li. For completeness, we present an argument in section 4.3, where we also prove some related stability properties for (C, δ, ϵ)-connectivity.

The work of Bate and Li provide a method to decompose a RNP-differentiability space (X, d, μ) into parts with asymptotic and non-homogeneous forms of Poincaré-inequalities, as well as a uniform and asymptotic form of Definition 3.1.2. In fact, Bate's and Li's results establishing an asymptotic version of connectivity on RNP-differentiability spaces motivated our connectivity condition. However, what was missing was a method to enlarge the pieces arising from their construction into PI-spaces, and recognizing that their condition was intimately connected to Poincaré-inequalities. This involves two steps. On the one hand, a new candidate space

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[1]The assumption of porous sets is somewhat technical and is similar to the discussion in [13].
[2]Recall the discussion after Definition 1.2.11. The subset V is equipped with the restricted measure and metric.

which is well-connected should be constructed and that satisfies the connectivity condition intrinsically. On the other hand, we need to show a Poincaré-inequality for the new space. Theorem 3.3.17 resolves the second objective.

The following theorem is used to construct the enlarged space. The terminology used is defined below.

Theorem 1.2.63. *Let $r_0 > 0$ be arbitrary. Assume (X, d, μ) is a metric measure space and $K \subset A \subset X$ and K compact. Assume further that X is (D, r_0)-doubling along A, A is uniformly $(\frac{1}{2}, r_0)$-dense in X along K, and A with the restricted measure and distance is locally $(C, 2^{-60}, \epsilon, r_0)$-connected[3] along K. There exists constants $\overline{C}, \overline{\epsilon}, \overline{D} > 0$, and a complete metric space $\overline{K}$ which is $\overline{D}$-doubling and $(\overline{C}, \frac{1}{2}, \overline{\epsilon}, r_0 2^{-20})$-connected, and an isometry $\iota \colon K \to \overline{K}$ which preserves the measure. In particular, the resulting metric measure space $\overline{K}$ is a PI-space.*

For an intuitive, and slightly imprecise, overview of the proof of this construction one can consider the case of a compact subset $K \subset \mathbb{R}^n$. Here K is well-connected when thought of as a subset of $\mathbb{R}^n$, but may not be intrinsically connected. To satisfy a local Poincaré-inequality the space must be locally quasiconvex. In order to make K locally quasiconvex we will glue a metric space T to it. Consider the set $A = N_1(K) = \{x \in \mathbb{R}^n | d(x, A) \leq 1\}$, and take a covering of $A \setminus K$ by Whitney-balls. The centers of these balls g_k, called gaps, correspond to vertices b_k of our graph. These vertices are called bridge points. Additionally, we define vertices corresponding to net-points of K. These resulting vertices are attached by edges to each other at roughly the same scales resulting in the metric space T. It can be attached to K since K naturally sits at its boundary. This

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[3]By an iteration argument similar to Theorem 1.2.40, or as presented in [12], the constant 2^{-60}-could be replaced by any $0 < \delta < 1$, but this would make our proof more technical and is unnecessary.

construction is analogous to that of a hyperbolic filling [21, Section 2] (see also a more recent presentation in English [19]).

4.1.1 Outline

In the next section, we introduce the relevant concepts and show Theorem 1.2.62 assuming Theorem 1.2.63. Following this, we give the thickening construction and prove Theorem 1.2.63. In the final section, we include a different proof that tangents of RNP-differentiability spaces are PI-spaces and that our connectivity condition is preserved under Gromov-Hausdorff-convergence.

4.1.2 Notational conventions

We will be studying the geometry of complete proper metric measure spaces (X, d, μ). Where not explicitly stated, all the measures considered in this chapter will be Radon measures.

Definition 4.1.1. A metric measure space (X, d, μ) is said to be asymptotically doubling if for almost every $x \in X$ we have

$$\limsup_{r \to 0} \frac{\mu(B(x, 2r))}{\mu(B(x, r))} < \infty. \tag{4.1.2}$$

We also define a relative version of doubling for subsets.

Definition 4.1.3. A metric measure space (X, d, μ) is said to be uniformly (D, r_0)-doubling along $S \subset X$ if for all $0 < r < r_0$ and any $x \in S$ we have

$$\frac{\mu(B(x, 2r))}{\mu(B(x, r))} \leq D. \tag{4.1.4}$$

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Definition 4.1.5. A set $S \subset X$ is called porous if there exists a c such that for every $x \in S$ and every $r > 0$ there exists a $y \in B(x, r)$ such that $B(y, cr) \cap S = \emptyset$. A set S is called σ-porous if there exist porous sets S_i (with possibly different constants c_i), such that

$$S = \bigcup_i S_i.$$

Definition 4.1.6. Let (X, d, μ) be a metric measure space and $A \subset X$ a positive measure subset. A point $x \in A$ is called an (ϵ, r_0)-density point of A if for any $0 < r < r_0$ we have

$$1 - \epsilon \leq \frac{\mu(B(x, r) \cap A)}{\mu(B(x, r))} \leq 1.$$

We say that A is uniformly (ϵ, r_0)-dense along $S \subset A$ if every point $x \in S$ is an (ϵ, r_0)-density point of A.

4.2   Proof of Main theorem

In order to state our main theorem, we need to define the relevant notions. We are interested in understanding when a given metric measure space is PI-rectifiable.

Definition 4.2.1. A metric measure space (X, d, μ) is PI-rectifiable if there is a decomposition into sets $U_i, N \subset X$ such that

$$X = \bigcup_i U_i \cup N$$

where $\mu(N) = 0$ and there exist isometric and measure preserving embeddings $\iota_i \colon U_i \to \overline{U}_i$ with $(\overline{U}_i, \overline{d}_i, \overline{\mu}_i)$ PI-spaces (with possibly very different constants).

We define a differentiability space as in [28, 85].

Definition 4.2.2. A metric measure space (X, d, μ) is called a differentiability space (of analytic dimension $\leq M$) if there exist $U_i \subset X$ and Lipschitz functions $\phi_i \colon U_i \to \mathbb{R}^{n_i}$ (with $n_i \leq M$) such that $\mu(N) = 0$,

$$X = \bigcup_i U_i \cup N,$$

and such that for any Lipschitz function $f \colon X \to \mathbb{R}$, for every i and μ-almost every $x \in U_i$, there exists a *unique* linear map $D^{\phi_i} f(x) \colon \mathbb{R}^{n_i} \to \mathbb{R}$ such that

$$f(y) = f(x) + D^{\phi_i} f(x)(\phi_i(y) - \phi_i(x)) + o(d(x, y)).$$

A stronger definition is obtained by assuming differentiability of Lipschitz functions with certain infinite dimensional targets [12].

Definition 4.2.3. A metric measure space (X, d, μ) is a RNP-differentiability space (of analytic dimension $\leq N$) if there exist $U_i \subset X$ and associated Lipschitz functions $\phi_i \colon U_i \to \mathbb{R}^{n_i}$ (with $n_i \leq N$) such that $\mu(N) = 0$,

$$X = \bigcup_i U_i \cup N,$$

and such that for any Banach space V with the Radon-Nikodym property and any Lipschitz function $f \colon X \to V$, for every i and μ-almost every $x \in U_i$ there exists a *unique* linear map $D^{\phi_i} f(x) \colon \mathbb{R}^{n_i} \to V$ such that

$$f(y) = f(x) + D^{\phi_i} f(x)(\phi_i(y) - \phi_i(x)) + o(d(x, y)).$$

Definition 4.2.4. A Banach space V has the Radon-Nikodym property if every Lipschitz function $f : [0, 1] \to V$ is almost everywhere differentiable.

For several equivalent and sufficient conditions, consult for example [116] or the references in [32].

Definition 4.2.5. A metric measure space (X, d, μ) is called asymptotically well-connected if for any $0 < \delta < 1$ and almost every point x the exists $r_x > 0$, $0 < \epsilon_x < 1$ and $C_x \geq 1$ such that for any $y \in X$ such that $d(x, y) < r_x$ the pair (x, y) is $(C_x, \delta, \epsilon_x)$-connected.

We can now state a general theorem that characterizes PI-rectifiable metric measure spaces. This is proven later in this section.

Theorem 4.2.6. *A complete metric measure space (X, d, μ) is a PI-rectifiable if and only if it is asymptotically doubling and asymptotically well-connected.*

Using this we can prove the PI-rectifiability result.

Theorem 1.2.62. *A complete metric measure space (X, d, μ) is a RNP-Lipschitz differentiability space if and only if it is PI-rectifiable and every porous set has measure zero.*

Proof of Theorem 1.2.62: By [12, Theorem 3.8] we have (up to a change in terminology and some refinements in their arguments) that a RNP-Lipschitz differentiability space is asymptotically doubling and asymptotically well-connected. For the converse, we refer to the main result in [33], where it is shown that a PI space is a RNP-Lipschitz differentiability space. It is then trivial to conclude that a PI-rectifiable space is also a RNP-Lipschitz differentiability space if every porous set has measure zero. See the discussion at the end of the introduction in [13] as well as the arguments in [14].

□

Before embarking on a proof of Theorem 4.2.6, we wish to motivate the used constructions. By using measure theory arguments, the asymptotic connectedness can be refined to produce subsets such that the space X satisfies some doubling and connectivity estimates uniformly along such sets. Then a general "thickening" argument is used, that improves a relative form of connectivity to an intrinsic form of connectivity. We remark, that there is no unique "thickening" constructing, but it is our belief that gluing a tree-like metric space is the easiest way to obtain the desired conclusions. When we construct T, we ought to keep the following perspectives in mind.

1. The resulting space needs to be a complete locally compact metric measure space. In particular, the added intervals need measures associated with them.

2. The measures of the added intervals need to be controlled by μ in order to preserve doubling.

3. Doubling needs to be controlled. We shouldn't add too many intervals at any given scale and location.

4. The intervals should themselves be "simple" PI-spaces, i.e. the measures should be related to Lebesgue measure.

5. A curve fragment in A between points in K should be replaceable, up to small measure, by a possibly somewhat longer curve fragment in the glued space.

A natural construction to obtain these goals arises from a modification of a so called hyperbolic filling. This construction first appeared in [21], and later in [19].

We will first indicate how Theorem 4.2.6 follows from Theorem 1.2.63, and then proceed to prove that Theorem in detail.

Proof of Theorem 4.2.6: By general measure theory arguments, we can construct sets $X = \bigcup_i V_i \cup N = \bigcup_{i,j} K_i^j \cup N'$, such that each V_i are measurable, $K_i^j \subset V_i$ are compact, $\mu(N) = \mu(N') = 0$ and the following uniform estimates hold.

- X is (D_i, r_i)-doubling along V_i.

- X is $(C_i, 2^{-60}, \epsilon_i, r_i)$-connected along V_i for some $\epsilon_i < 1$.

- V_i is $(1 - \epsilon_i/2, r_j^i)$-dense in X along K_i^j.

For the definitions of these concepts see the beginning of Section 3.2.

Since the set V_i is uniformly $(1 - \epsilon_i/2, r_i^j)$-dense in X along K_i^j, directly we obtain that V_i with its restricted measure and metric is $(C_i, 2^{-60}, \epsilon_i/2, \min(r_i, r_i^j)/C_i)$-connected along K_i^j. To see this, verify the definition of 3.1.2 by adjoining the complement $X \setminus V_i$ to any set E considered.

Theorem 1.2.63 can now be applied to $A = V_i$, $K = K_i^j$ and $\epsilon = \epsilon_i/2, r_0 = \min(r_i, r_i^j)/C_i = r_0^{ij}, C = C_i$ and $D = D_i$ to obtain isometric and measure preserving embeddings $\iota \colon K_i^j \to \overline{K}_i^j$ to metric measure spaces $(\overline{K}_i^j, d_i^j, \mu_i^j)$. Further, by the same theorem, there exist positive constants $\overline{D}_{i,j}, \overline{C}_{i,j}$ and $\overline{\epsilon}_{i,j}$ such that the metric measure spaces $(\overline{K}_i^j, d_i^j, \mu_i^j)$ are locally $(\overline{D}_{i,j}, r_0^{i,j} 2^{-20})$-doubling and locally $(\overline{C}_{i,j}, 2^{-1}, \overline{\epsilon}_{i,j}, r_0^{i,j} 2^{-20})$-connected. Thus, by the main Theorem 3.3.17 each $\overline{K}_i^j$ is a PI-space. This completes the proof of PI-rectifiability.

$\square$

4.2.1   Construction

Continue next with the proof of Theorem 1.2.63. First, we present the construction, which is followed by a separate subsection containing the proofs of the relative connectivity and doubling properties.

"Thickening" construction for Theorem 1.2.63: Assume for simplicity that $D \geq 2$. Re-scale so that $r_0 = 2^{20}$. We will define $\overline{K} = K \cup T$, where T arises as the metric space induced by a metric graph (V, E), and K naturally sits at the boundary of T. The vertices will arise from two types of points, net points of K and gap points that correspond to center points in $A \setminus K$.

First define the 2^{-j}-nets N_j of K for $j \geq 0$, i.e. maximal collections of points such that $d(n, m) \geq 2^{-j}$ for $n, m \in N$. In order to make some notation below consistent define $N_{-1} = \emptyset$. Define these inductively such that $N_j \subset N_{j+1}$, and define $N = \bigcup_j N_j$ the collection of all net points. Further, define the scale function as $\mathbf{Sc}_N(n) = 2^{-i}$ if $n \in N_i \setminus N_{i-1}$

Using Vitali covering Lemma [55, 135], choose "gap" points $g_m^k \in (N_1(K) \setminus K) \cap A$, indexed by $(k, m) \in U \subset \mathbb{N}_0 \times \mathbb{N}_0$, such that

$$(N_1(K) \setminus K) \cap A \quad \subset \quad \bigcup_{(k,m) \in U} B\left(g_k^m, 2^{-k-10}\right), \qquad (4.2.7)$$

$$2^{-k-1} < \quad d(g_k^m, K) \quad \leq 2^{-k} \qquad (4.2.8)$$

and for $(k, m) \neq (k', m')$ with $(k, m), (k', m') \in U$

$$B\left(g_k^m, 2^{-k-15}\right) \cap B\left(g_{k'}^{m'}, 2^{-k'-15}\right) = \emptyset. \qquad (4.2.9)$$

Recall that $N_1(K) = \{x \in X | d(x, K) < 1\}$. This type of covering is also known as a Whitney-covering. Given such a covering, define the gap points at scale 2^{-k} by $G_k = \{g_k^m | (k, m) \in U\}$ and $G = \bigcup_k G_k$ the collection of all gap points. We will define the function $\mathbf{Sc}_G(g) = 2^{-k}$ if $g \in G_k$.

Define the vertex sets as $V = V_G \cup V_N$, where V_G arises from G and V_N arises from N. Vertices will be defined as pairs (x, r) with $x \in X$ and $r > 0$. First take

$$V_G = \left\{(g, 2^{-k}) | g \in G_k\right\}.$$

These will also be referred to as bridge-points because they are used to repair the gap corresponding to the center g at scale 2^{-k}.

Next, define for each $n \in N$ the maximal scale at which there is a near-by gap point.

$$l(n) = \max\left\{2^{-k} \, | \exists g \in G_k : d(g, n) \leq 2^{4-k} \text{ and } \mathbf{Sc}_G(g) = 2^{-k} \leq \mathbf{Sc}_N(n)\right\}.$$

$$(4.2.10)$$

If there is no such point, we will define $l(n) = 0$. We define the vertex set arising from the net points as

$$V_N = \left\{(n, 2^{-k}) | n \in N, 2^{-k} \leq l(n)\right\}.$$

Define the edge set E as follows. Two distinct vertices $(x, r), (y, s) \in V$ are

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connected[4], i.e. $e = ((x, r), (y, s)) \in E$ if

$$d(x, y) \leq 2^4(r + s), \tag{4.2.11}$$

and

$$\frac{1}{2} \leq r/s \leq 2. \tag{4.2.12}$$

We define a simplicial complex T with edges corresponding to $e \in E$ and vertices corresponding to $v \in V$. In this geometric realization use I_e to represent the interval corresponding to e. For each $v \in V$ we represent the corresponding point in T with the same symbol. The simplicial complex becomes a metric space by declaring the length of the edge $e = ((a, s), (b, t))$ to be

$$|e| = 2^4(s + t). \tag{4.2.13}$$

A metric d_T is induced by the path metric on the simplicial complex. Since T may not be connected, some pairs of points $v, w \in T$ may have $d_T(v, w) = \infty$.

When a point x lies on an interval I_e, we will often abuse notation and say $x \in e$. Also, we will use the word edge to refer either to the symbol e associated to it, or the interval I_e in the geometric realization. Define the set which will be our metric space as $\overline{K} = K \cup T$. We define a symmetric function δ defined on a subset of $\overline{K} \times \overline{K}$ and the metric $\overline{d}$ on $\overline{K}$ to be the maximal metric subject to

- $\forall x, y \in K : \overline{d}(x, y) \leq d(x, y) = \delta(x, y)$

---

[4]A minor technical point is that we use tuples. Thus, really the resulting space is a directed graph. We could also use undirected graphs, but it will simplify notation below to allow both edges.

- $\forall n \in N, (n,s) \in V_N : \overline{d}(n,(n,s)) \leq 3 \cdot 2^4 s = \delta(n,(n,s))$

- $\forall v, w \in T : \overline{d}(v,w) \leq d_T(v,w) = \delta(v,w)$

The distance can be given explicitly by a minimization over discrete paths. Let $x, y \in \overline{K}$ and $\sigma = (\sigma_0, \ldots, \sigma_m)$ be a sequence of points in $\overline{K}$. The variable m is called the length of the path. We call such a sequence a discrete path in $\overline{K}$, and say that it connects x to y, if $\sigma_0 = x, \sigma_m = y$. We call it admissible, if for each consecutive pair σ_i, σ_{i+1}, either they both lie in T or K, or one of them is equal to n, for some $n \in N$, and the other is equal to (n,s) for $(n,s) \in V_N$. In other words, we require $\delta(\sigma_i, \sigma_{i+1})$ to be defined by one of the cases above. Denote by $\Sigma_{x,y}^n$ the space of all admissible discrete paths of length n that connect x to y. With these definitions, the distance becomes

$$\overline{d}(x,y) = \inf_n \inf_{\sigma \in \Sigma_{x,y}^n} \sum_{i=0}^{n-1} \delta(\sigma_i, \sigma_{i+1}). \qquad (4.2.14)$$

A measure on each interval $I_e = [0, |e|]$, which is associated to the edge $e = ((x,r),(y,s))$, is defined as weighted Lebesgue measure:

$$\mu_e = \frac{\mu(B(x,r)) + \mu(B(y,s))}{|e|} \lambda. \qquad (4.2.15)$$

The total measure is $\mu_T = \sum_e \mu_e$, where each measure μ_e is extended to be zero on the complement of I_e, and the measure on the new space $\overline{K}$ is defined as μ_T on T and $\mu|_K$ on K. In other words

$$\overline{\mu} = \mu_T + \mu|_K.$$

4.2.2 Proof that $(\overline{K}, \overline{d}, \overline{\mu})$ is a PI-space

Our goal is to show that the space constructed in the previous subsection satisfies the properties from Theorem 1.2.63. This involves some preliminary lemmas.

Definition 4.2.16. A metric space (X, d) is said to be uniformly (L, r_0)-perfect if for any $B(x, r)$ with $0 < r < r_0$ the following holds

$$B(x, r_0) \setminus B(x, r) \neq \emptyset \implies B(x, r) \setminus B(x, r/L) \neq \emptyset.$$

The space X is said to be uniformly (L, r_0)-perfect along a subset $S \subset X$ if the same holds for any $x \in S$ and $0 < r < r_0$.

Lemma 4.2.17. *Let $r_0, C, \epsilon > 0$ be arbitrary. If A is a metric measure space and A is $(C, \delta, \epsilon, r_0)$-connected along $K \subset A$ for some $0 < \delta < \frac{1}{8}$, then A is uniformly $(7/5, r_0)$-perfect along K[5]*

Proof: Choose an arbitrary $x \in K$ and $0 < r < r_0$. Assume $B(x, r_0) \setminus B(x, r) \neq \emptyset$. Let $s = \inf_{y \in A \setminus B(x, 5r/7)} d(x, y)$. By assumption $s < r_0$. If $s < r$, we are done. For the sake of contradiction, assume that $s \geq r$. Choose y such that $d(x, y) < 8s/7$ and $d(x, y) < r_0$. By the definition of s we know that $B(x, s) \setminus B(x, 6s/7)$ is empty. Then, by the connectivity assumption (and choosing $E = \emptyset$), there is a 1-Lipschitz curve fragment $\gamma \colon K' \to A$ connecting x and y with $|\mathrm{Undef}(\gamma)| < \frac{1}{8} d(x, y)$. Assume $a = \sup_{t \in K', \gamma(t) \in B(x, 6s/7)} t$ and $b = \inf_{a < t \in K', \gamma(t) \in A \setminus B(x,s)}$. Then

$$d(x, y)/8 < s/7 \leq d(\gamma(a), \gamma(b)) \leq |b - a|.$$

---

[5]Proving this would be easier if the space was quasiconvex. However, this is not necessarily true when the connectivity assumption only holds in a relative sense. In Lemma 3.3.4 we assumed stronger connectivity.

Further, the interval $(a, b) \subset \mathbb{R} \setminus K'$. Namely, if $t \in K' \cap (a, b)$, we have $\gamma(t) \in B(x, s)$ (by definition of b), but then automatically $\gamma(t) \in B(x, 6s/7)$, which contradicts the definition of a. Then we conclude that $|b-a| \le |\text{Undef}(\gamma)| < d(x, y)/8$, and this is a contradiction.

$\square$

This lemma is mainly used to give upper bounds for volumes in a doubling metric measure space.

Lemma 4.2.18. *Let $r_0, D > 0$ be arbitrary. Assume that X is a complete metric space, X is (D, r_0)-doubling along $A \subset X$ and A is uniformly $(7/5, r_0)$-perfect along a set $K \subset A$. Then if $x \in K$, $0 < r < r_0$ and $A \cap B(x, r_0) \setminus B(x, r) \neq \emptyset$,*

$$\mu(B(x, r/2)) \le (1 - \sigma)\mu(B(x, r)) \tag{4.2.19}$$

for $\sigma = \frac{1}{D^4}$.

Proof: By the previous lemma (applied for $7r/8$) and the assumption there is a point $y \in A \cap B(x, 7r/8) \setminus B(x, 5r/8)$. For such a y we have $B(y, r/8) \subset B(x, r) \setminus B(x, r/2)$. Also, by doubling

$$\mu(B(y, r/8)) \ge \frac{\mu(B(x, r))}{D^4},$$

which gives

$$\mu(B(x, r/2)) \le \mu(B(x, r)) - \mu(B(y, r/8)) \le \left(1 - \frac{1}{D^4}\right)\mu(B(x, r)).$$

$\square$

We will also need a lemma concerning concatenating curve fragments.

Lemma 4.2.20. *Assume (X, d, μ) is a (D, r_0)-doubling metric space. If $1 > \epsilon, \delta > 0$, $L, C \geq 1$ are fixed and $p_i \in X$, for $i = 1, \ldots, n$, are points such that each pair (p_i, p_{i+1}) is (C, δ, ϵ)-connected and $\sum_{i=1}^{n-1} d(p_i, p_{i+1}) \leq L d(p_1, p_n)$, then (p_1, p_n) is*

$$(LC, 2L\delta, \epsilon D^{\log_2(\delta) - \log_2(L) - \log_2(n) - \log_2(C) - 6})\text{-connected.}$$

Proof: Denote $r = d(p_1, p_n)$, and assume $E \subset B(p_1, LCr)$ with

$$\mu(E \cap B(p_1, LCr)) \leq \epsilon D^{\log_2(\delta) - \log_2(L) - \log_2(n) - \log_2(C) - 6} \mu(B(p_1, LCr)).$$

Define $l(1) = 0$ and $l(j) = \sum_{i=1}^{j-1} d(p_i, p_{i+1})$ for $j = 2, \ldots, n$. Also, define the set of intervals $\mathcal{I} = \{(Cl(i), Cl(i+1)) | i = 1, \ldots, n-1\}$ and $\mathcal{G} = \{I \in \mathcal{I} | |I| \geq \frac{\delta r}{n}\}$. For each $I = (Cl(i), Cl(i+1)) \in \mathcal{G}$ we have $d_i = d(p_i, p_{i+1}) \geq \frac{\delta r}{nC}$. In particular, by doubling

$$\mu(E \cap B(p_i, Cd_i)) \leq \epsilon D^{\log_2(\delta) - \log_2(L) - \log_2(n) - \log_2(C) - 6} \mu(B(p_1, LCr)) \leq \epsilon \mu(p_i, Cd_i).$$

Thus, we can define $\gamma_I \colon K_I \to X$ to be a 1-Lipschitz curve fragment connecting p_i to p_{i+1} with $K_I \subset I$, $\mathrm{len}(\gamma_I) \leq |I| = Cd_i$, $|\mathrm{Undef}(\gamma_I)| < \delta d_i$ and $\gamma_I(I) \cap E = \emptyset$. Further, assume by a slight dilation that $\min(K_I) = \min(I), \max(K_I) = \max(I)$. Define $K = \{l(j)\} \cup \bigcup_{I \in \mathcal{G}} K_I$. Next, define a 1-Lipschitz curve fragment $\gamma \colon K \to X$

by setting $\gamma(l(i)) = p_i$ and $\gamma(t) = \gamma_I(t)$ for $t \in I \in \mathcal{G}$. Clearly $\operatorname{len}(\gamma) \leq \max(K) \leq LCr$, and $\operatorname{Undef}(\gamma) \subset \bigcup_{I \in \mathcal{I} \setminus \mathcal{G}} I \cup \bigcup_{I \in \mathcal{G}} \operatorname{Undef}(\gamma_I)$. Thus,

$$|\operatorname{Undef}(\gamma)| \leq n \frac{\delta d(x,y)}{n} + \delta \sum_{i=1}^{n-1} d_i < 2L\delta r.$$

Since p_i might belong to E, we might need to remove small neighborhoods of these points to satisfy the third condition in 3.1.2. This modification of γ is trivial.

$\square$

Further, since our space $\overline{K}$ arises by gluing a tree to K, the following lemma is useful.

Lemma 4.2.21. *Assume* (X, d, μ) *is a* (D, r_0)-*doubling metric space, and* $I \subset X$ *is isometric to a bounded interval in* $\mathbb{R}$. *Assume further that the restricted measure is given as* $\mu|_I = c\lambda$ *where* λ *is the induced Lebesgue measure on* I *and* $c > 0$ *is some constant. Then for any* $\delta > 0$ *and any* $x, y \in I$ *we have that* (x,y) *is* $(1, \delta, \delta(2D)^{-2})$-*connected.*

Proof: Let $x, y \in I$. Denote by $r = d(x,y)$ and by z the midpoint of x and y on the interval I. Take an arbitrary Borel set E with

$$\frac{\mu(B(x,r) \cap E)}{\mu(B(x,r))} \leq \delta/(2D)^2$$

We will connect the pair of points x, y by the geodesic segment $\gamma \colon J \to X$ where $J \subset I$ is the sub-interval defined by x and y. Note that $J = B(z, r/2)$. The curve γ may intersect with E, but

$$
\begin{aligned}
|\gamma^{-1}(E)| &\leq \lambda(J \cap E) \leq \frac{1}{c}\mu(J \cap E) \\
&\leq \lambda(J)\frac{\mu(J \cap E)}{\mu(J)} \leq r\frac{\mu(E \cap B(x,r))}{\mu(B(z,r/2))} \\
&\leq rD^2\frac{\mu(E \cap B(x,r))}{\mu(B(x,r))} \leq rD^2\delta/(2D)^2 = \delta r/2.
\end{aligned}
$$

Thus, by restricting γ to an almost full measure sub-set in the complement of the set $\gamma^{-1}(E)$, we obtain the desired curve fragment.

$\square$

Occasionally, we will need to vary the constant C in Definition 3.1.2. Thus, we use the following.

Lemma 4.2.22. *Assume (X, d, μ) is a (D, r_0)-doubling metric space and $0 < \delta, \epsilon$, $C \geq 1$ and $K \geq 0$ given constants. If $(x, y) \in X \times X$ with $d(x, y) \leq r_0 2^{-K}/C$ is (C, δ, ϵ)-connected then it is also $(2^K C, \delta, \epsilon D^{-K-1})$-connected.*

Proof: Denote by $r = d(x, y)$. Take an arbitrary Borel set E with

$$
\frac{\mu(B(x, 2^K Cr) \cap E)}{\mu(B(x, 2^K Cr))} \leq \epsilon D^{-K-1}.
$$

Then doubling gives also

$$
\frac{\mu(B(x, Cr) \cap E)}{\mu(B(x, Cr))} \leq \epsilon.
$$

Thus, the result follows from the assumption and Definition 3.1.2.

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$\square$

Proof that the construction in Subsection 4.2.1 works for Theorem 1.2.63:

Preliminaries: Continue using the notation of Subsection 4.2.1. For future reference, we compute some basic estimates and define some related notation. Define the scale function $\mathbf{Sc}\colon V \to \mathbb{R}$ which is given by $\mathbf{Sc}((x,r)) = r$ and location function $\mathbf{Loc}\colon V \cup K \to K$ which assigns for $(x,r) \in V$ the value $\mathbf{Loc}((x,r)) = x$ and restricts to identity on K. For a vertex $v = (x,r) \in V$ we denote the set $E_v = \{e \in E | e \text{ is incident to } v\}$.

Lemma 4.2.23. *For all $v \in V$ we have*

$$|E_v| \leq 4D^{25}. \tag{4.2.24}$$

To see this consider $V_v^N = \{w | w \in V_N, (v,w) \in E\}$ and $V_v^G = \{w | w \in V_G, (v,w) \in E\}$. Then $|E_v| \leq 2|V_v^N| + 2|V_v^G|$. For $w \in V_v^G$, then $\mathbf{Sc}(w) \geq r/2$, and $d(\mathbf{Loc}(w), \mathbf{Loc}(v)) \leq 2^6 r$. In particular, by the disjointedness property in Equation (4.2.9), the points $\mathbf{Loc}(w)$ have pairwise distance at least a $r2^{-15}$ in $B(\mathbf{Loc}(v), 2^6 r) \cap A$. Thus, their number is bounded by D^{25}, i.e. $|V_v^G| \leq D^{25}$.

Similarly, if $w \in V_v^N$ we have $\mathbf{Sc}(w) \geq r/2$. In particular, $l(\mathbf{Loc}(w)) \geq r/2$. Thus, the pairwise distances of distinct $\mathbf{Loc}(w)$ are at least $r/2$. Also, $d(\mathbf{Loc}(w), \mathbf{Loc}(v)) \leq 2^6 r$. The number of such w is bounded by $|V_v^N| \leq D^9$. We get by summing up the estimates for V_v^G and V_v^N that

$$|E_v| \leq 4D^{25}.$$

Lemma 4.2.25. *The following estimates are true on* $\overline{K}$.

1. *For every* $x, y \in K \subset \overline{K}$: $\overline{d}(x, y) = d(x, y)$.

2. *For every* $e \in E$ *and every* $x, y \in I_e \subset T$ *we have* $\overline{d}(x, y) = d_T(x, y)$.

3. *For every* $(x, r) \in V$ *there is a* $n \in N$ *such that* $e = ((n, r), (x, r)) \in E$. *In particular,* $\overline{d}((x, r), n) \leq 2^6 r$.

4. *If* $(x, r) \in V$ *and* $y \in K$, *then* $\max\{r, d(x, y)\} \leq \overline{d}((x, r), y) \leq d(x, y) + 2^8 r$.

5. *If* $(x, r), (y, s) \in V$ *are distinct, then* $\max\{d(x, y), r + s\} \leq \overline{d}((x, r), (y, s)) \leq d(x, y) + 2^4(r + s)$.

6. *If* $x \in e = (v, w) \in E$ *and* $y \in \overline{K} \setminus e$, *then* $\overline{d}(x, y) = \min\{\overline{d}(x, v) + \overline{d}(v, y), \overline{d}(x, w) + \overline{d}(w, y)\}$.

7. *If* $v \in V$, *and* $e \in E_v$, *then* $|e| \leq 2^7 \boldsymbol{Sc}(v)$ *and* $\mu(B(\boldsymbol{Loc}(v), \boldsymbol{Sc}(v))) \leq \overline{\mu}(I_e) \leq 2D^7 \mu(B(\boldsymbol{Loc}(v), \boldsymbol{Sc}(v)))$.

8. *For any* $n \in N$ *and any* $0 < r \leq l(n)$ *we have*

$$\sum_{2^l \leq r} \sum_{e \in E_{(n,2^l)}} \overline{\mu}(I_e) \leq 2^{20} D^{37} \mu(B(n, 2^l)).$$

9. *There is a function* $\rho\colon N \to G \cup N$ *such that* $\rho(n) = n$ *if* $l(n) = 0$, *and if* $l(n) > 0$, *then* $\rho(n) \in G_{l(n)}$ *and*

$$d(\rho(n), n) \leq 2^4 l(n).$$

Further, for any $g \in G$ we have $\rho^{-1}(g) \leq D^{25}$.

Proof: We will proceed in numerical order.

Estimate 1: Take arbitrary $x, y \in K$. It is obvious that $\bar{d}(x, y) \leq d(x, y)$. For an arbitrary $\epsilon > 0$ we can find a $n \in \mathbb{N}$ and $\sigma \in \Sigma_{x,y}^n$ such that

$$\bar{d}(x, y) + \epsilon \geq \sum_i \delta(\sigma_i, \sigma_{i+1}).$$

We can assume by possibly making the sum on the right smaller that $\sigma_i \in K$ or $\sigma_i \in V \subset T$ for all $i = 0, \ldots, n$. Thus, there is a function $f \colon \{0, \ldots, n\} \to K$, given by $f(i) = \sigma_i$ if $\sigma_i \in K$ and $f(i) = \mathbf{Loc}(\sigma_i)$ if $\sigma_i \in V$. By the definition of δ preceding (4.2.14) we have for $i = 0, \ldots, n-1$

$$d(f(i), f(i+1)) \leq \delta(\sigma_i, \sigma_{i+1}).$$

By the triangle inequality

$$\bar{d}(x, y) + \epsilon \geq \sum_i d(f(i), f(i+1)) \geq d(x, y).$$

Since $\epsilon > 0$ is arbitrary $\bar{d}(x, y) \geq d(x, y)$, which shows $\bar{d}(x, y) = d(x, y)$.

Estimate 2: The proof, that we have for any points $x, y \in T$ on a common edge $e \in E$ the equality $d_T(x, y) = \bar{d}(x, y)$, is a trivial application of the definition of δ and observing that a discrete path that does not directly connect the points x, y in e will pass through an edge adjacent to both end points of e, or two separate

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edges adjacent to the end points of e. The length of such a path is larger than $|e| \geq d_T(x, y)$.

Estimate 3: Next, take an arbitrary $(x, r) \in V$. We wish to find a near-by $n \in N$. Either $x \in N$ or $x \in G$. In the first case define $x = n$, and by the definition of $\bar{d}$ we have

$$\bar{d}(n, (x, r)) \leq 3 \cdot 2^4 r \leq 2^6 r.$$

Assume that $x = g \in G$. Then by (4.2.8) there is a $k \in K$ such that $d(x, k) \leq r$. Let $n \in N$ be a net point closest to k with $\mathbf{Sc}(n) \leq r \leq 2\mathbf{Sc}(n)$. Then $d(n, k) \leq r$. Thus, by the triangle inequality $d(g, n) \leq 2r \leq 2^2 \mathbf{Sc}(n)$, and by (4.2.10) we have $l(n) \geq r$. Then both (n, r) and (g, r) are vertices in V joined by an edge. Further, $\bar{d}((n, r), (g, r)) = 2^5 r$ by Estimate 2 above and (4.2.13). By the triangle inequality and what we just observed $\bar{d}(n, (g, r)) \leq 2^6 r$.

Estimate 4: Take a $y \in K$ and $(x, r) \in V$. Choose $n \in N$ such that $\bar{d}((x, r), n) \leq 2^6 r$. We have $d(y, n) \leq d(x, n) + d(y, x) \leq 2^6 r + d(y, x)$. Thus, $\bar{d}((x, r), y) \leq 2^7 r + d(x, y)$.

In order to obtain the desired lower bound, consider the height function given by $h \colon \overline{K} \to \mathbb{R}$ and defined as $h(x) = 0$ for $x \in K$ and by $h(v) = r$ for $v = (x, r) \in V$. On each edge extend f linearly. Now for any $x, y \in \overline{K}$ for which δ was defined (see discussion preceding (4.2.14)) we get $|h(x) - h(y)| \leq \delta(x, y)$. Thus, for any discrete path σ connecting x to y we obtain

$$|h(x) - h(y)| \leq \sum_i \delta(\sigma_i, \sigma_{i+1}).$$

Taking an infimum over σ we get $|h(x) - h(y)| \leq \overline{d}(x, y)$. A particular case of this gives when $(x, r) \in V$ and $y \in K$

$$r = |h((x, r)) - h(y)| \leq \overline{d}((x, r), y).$$

The lower bound $d(x, y) \leq \overline{d}((x, r), y)$ is proven similarly to the corresponding one in Estimate 5.

Estimate 5: Let $(x, r), (y, s) \in V$ be distinct. Using Estimate 3 we can find $n_x, n_y \in K$ such that $\overline{d}((x, r), n_x) \leq 2^6 r$ and $\overline{d}((y, s), n_y) \leq 2^6 s$. Further, by the calculations for Estimate 3 we get $d(n_x, x) \leq 2r$ and $d(n_y, y) \leq 2r$. A trivial application of the triangle inequality then gives $\overline{d}((x, r), (y, s)) \leq 2^8 r + d(x, y)$. Next consider the lower bound. Let σ be any discrete path that passes through vertices in V and points in K. The discrete path must traverse an edge adjacent to (x, r), and one adjacent to (y, s). These edges might be the same, or different, but in either case one can verify that their total length is at least $r + s$, and thus $r + s \leq \overline{d}((x, r), (y, s))$. Next, with the triangle inequality we obtain

$$d(x, y) \leq \sum_i d(\mathbf{Loc}(\sigma_i), \mathbf{Loc}(\sigma_{i+1})) \leq \sum_{i=0}^{n-1} \delta(\sigma_i, \sigma_{i+1}).$$

Taking the infimium gives $d(x, y) \leq \overline{d}(x, y)$, which completes the proof.

Estimate 6: Trivial from the definition in (4.2.14).

Estimate 7: These estimates follow directly from doubling and (4.2.11) and (4.2.12).

Estimate 8: Take an arbitrary $n \in N$ and $r \leq l(n)$. Let $r \leq L = 2^k \leq l(n)$ such that $L \leq 2r$. For any $s \leq l(n)$ and any $e \in E_{(n,s)}$ using Estimate 6 of this lemma we obtain

$$
\begin{aligned}
\mu_e(I_e) &= \mu(B(n,s)) + \mu(B(y,t)) \\
&\leq 2D^7 \mu(B(n,s)).
\end{aligned}
\tag{4.2.26}
$$

Apply Lemma 4.2.23 to get

$$
\begin{aligned}
\sum_{2^l \leq r} \sum_{e \in E_{(n,2^l)}} \overline{\mu}(I_e) &\leq \sum_{2^l \leq L} \sum_{e \in E_{(n,2^l)}} \overline{\mu}(I_e) \\
&\leq \sum_{2^l \leq L} 8D^{32} \mu(B(n,2^l))
\end{aligned}
$$

The last sum can be estimated using Lemma 4.2.49 to get

$$
\sum_{2^l \leq L} 8D^{32} \mu(B(n,2^l)) \leq 2^{20} D^{36} \mu(B(n,L)) \leq 2^{20} D^{37} \mu(B(n,r)).
$$

The factor 2^{10} arises from the fact that the non-emptyness assumption of Lemma 4.2.49 can be guaranteed only for $s \leq L 2^{-4} \leq l(n) 2^{-4}$.

Estimate 9: In the case $l(n) = 0$, define $\rho(n) = n$. By the definition of $l(n)$ in (4.2.10), for each $n \in N$ such that $l(n) > 0$ we can find a $g \in G_{l(n)}$ such that $d(g, n) \leq 2^4 l(n)$. In this case, define $\rho(n) = g$ by choosing one of them. This defines a function $\rho \colon N \to G \cup N$. Next, if $g \in G$ we can use the definition of $l(n)$ to obtain two properties. Firstly, for any $n \in \rho^{-1}(g)$ we have the estimate $d(g, n) \leq 2^4 \mathbf{Sc}(g)$. Secondly, if $n, m \in \rho^{-1}(g)$ are distinct then $d(n, m) \geq \mathbf{Sc}(g) 2^{-15}$ (by (4.2.9)). Thus, by doubling their total number can be bounded by doubling as $|\rho^{-1}(g)| \leq D^{25}$.

$\square$

Next, we show completeness by taking an arbitrary Cauchy sequence x_i and finding a limit point. If infinitely many of the x_i lie in K, or a single edge I_e, the result follows trivially. Thus, we can assume by passing to a sub-sequence that each $x_i \in e_i = ((a_i, r_i), (b_i, s_i))$ for distinct edges $e_i \in E$. We can assume $r_i \to 0$, since by the compactness of K we obtain that for any given $\epsilon > 0$ there are only finitely many edges with $r_i > \epsilon$. By estimate (4.2.8) we have $d(a_i, K) \leq r_i$, and thus by Estimate 4 in 4.2.25 there is an $k \in K$ such that

$$\bar{d}((a_i, r_i), k_i) \leq 2^9 r_i.$$

On the other hand by (4.2.11) and the second estimate in Lemma 4.2.25 we get

$$\bar{d}((a_i, r_i), x_i) \leq 2^9 r_i + \bar{d}((a_i, r_i), (b_i, s_i)) \leq 2^9 r_i + d(a_i, b_i) + 2^8 (r_i + s_i) \leq 2^{15} r_i.$$

In particular, $\overline{d}(x_i, k_i) \leq 2^{20} r_i$. Thus, we can conclude that k_i is also a Cauchy sequence in K and has a limit point k_∞. It is now easy to see that $\lim_{i \to \infty} x_i = k_\infty$.

Our desired metric measure space is $(\overline{K}, \overline{d}, \overline{\mu})$. The remainder of the proof shows that it is a PI-space. We have already observed that $\overline{K}$ is complete. Since K and T are totally bounded with respect to $\overline{d}$ it is not hard to see that $\overline{K}$ is also totally bounded and thus compact. The measure is a sum of finite Radon measures and thus as long as it's bounded it will itself be a Radon measure. This boundedness follows from estimates below. It remains to show the doubling and connectivity properties. The last step is to apply Theorem 3.3.17 to conclude that the resulting space is a PI-space.

In order to distinguish between metric and measure notions on $\overline{K}$ and those on the original space X we will often use a line above the symbol. For example for $x \in K$, the metric ball in X is denoted $B(x, r)$, and the ball in $\overline{K}$ is denoted $\overline{B}(x, r)$. There is no risk of confusing this with the closure of the ball because we will not be using that in any of the arguments below.

Doubling: The doubling condition 1.2.14 depends on two parameters (x, r) and we need to check three cases depending on the location and scale. This is done by estimating the volume of various balls from above and below.

Case 1: $x \in K$ **and** $0 < r < \frac{1}{2}$**:**

- Lower bound for $x \in K, 0 < r < 1$: We will use two disjoint subsets of $\overline{B}(x, r)$: B_K (part in K) and B_T (corresponding to a portion of the glued intervals). In precise terms we set

$$\overline{B}_K = \overline{B}(x, r) \cap K$$

and

$$\overline{B}_T = \bigcup_{\substack{v \in V_G, e \in E_v \\ \overline{d}(v,x) < r2^{-10}}} I_e.$$

If $y \in \overline{B}_T$, then there is a $v \in V$ such that $y \in e \in E_v$ and $\overline{d}(v, x) < r2^{-10}$. by Cases 3, 6 and 7 of Lemma 4.2.25 we have $\overline{d}(x, y) \leq |e| + \overline{d}(x, v) \leq r2^{-3} + r2^{-10} < r$. Thus

$$\overline{B}_K \cup \overline{B}_T \subset \overline{B}(x, r). \tag{4.2.27}$$

The sets $\overline{B}_K$ and $\overline{B}_T$ are disjoint, and we can estimate the total volume from below by estimating each of them individually and summing the estimates.

$$\overline{\mu}(\overline{B}_K) = \mu(B(x, r) \cap K) \tag{4.2.28}$$

The volume of $\overline{B}_T$ can be estimated using Case 7 and Case 4 of Lemma 4.2.25.

$$\overline{\mu}(\overline{B}_T) \geq \bigcup_{\substack{v=(g,s) \in V_G, e \in E_v \\ \overline{d}(v,x) < r2^{-10}}} \mu_e(I_e) \tag{4.2.29}$$

$$\geq \sum_{\substack{v=(g,s) \in V_G, e \in E_v \\ d(g,x) < r2^{-20}}} \mu(B(g, s)) \tag{4.2.30}$$

$$\geq \frac{1}{2}\mu(B(x, r2^{-26}) \cap A \setminus K). \tag{4.2.31}$$

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In the last line we used the covering property

$$B(x, r2^{-26}) \cap A \setminus K \subset \bigcup_{\substack{g \in G \\ d(x,g) \leq r2^{-20}}} B(g, \mathbf{Sc}(g)2^{-10})).$$

To see this take an arbitrary $z \in B(x, r2^{-13}) \cap A \setminus K$. Then by the Whitney covering property (4.2.7) there is a $g \in G$ such that

$$z \in B(g, \mathbf{Sc}(g)2^{-10}).$$

However, $d(g, x) \geq \mathbf{Sc}(g)2^{-1}$ by (4.2.8) and $d(g, x) \leq r2^{-26} + \mathbf{Sc}(g)2^{-10}$ by the triangle inequality. Thus, we get $\mathbf{Sc}(g) \leq r2^{-25}$ and $d(g, x) \leq r2^{-25}$. This is sufficient to verify the previous inclusion.

We obtain using estimates (4.2.28) and (4.2.31) that

$$\begin{aligned} \bar{\mu}(\overline{B}(x, r)) &\geq \bar{\mu}(B_K) + \bar{\mu}(B_T) \\ &\geq \frac{1}{2}\mu(B(x, r2^{-26}) \cap A \setminus K) + \mu(B(x, r2^{-26}) \cap A \cap K) \\ &\geq \frac{1}{2}\mu(B(x, r2^{-26}) \cap A) \geq \frac{1}{D^{26}4}\mu(B(x, r)). \qquad (4.2.32) \end{aligned}$$

The uniform density of A for $x \in K$ was used on the last line.

- Upper bound for $x \in K, 0 < r < 1$: Define the following sets of edges:

$$E_2 = \{e \in E | \exists v = (n, s) \in V_N, e \in E_v, d(n, x) \leq r, s \leq l(n) \leq r\},$$

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$$E_3 = \{e \in E | \exists v = (n, s) \in V_N, e \in E_v, d(n, x) \le r, l(n) \ge r \ge s\},$$

$$E_4 = \{e \in E | \exists v = (g, s) \in V_G, e \in E_v, d(g, x) \le r\}.$$

By the definition of the space and simple distance estimates from Cases 1–7 in Lemma 4.2.25, we can decompose the set into pieces as follows.

$$\overline{B}(x, r) \subset A_1 \cup A_2 \cup A_3 \cup A_4,$$

with $A_1 = \overline{B}(x, r) \cap K$,

$$A_2 = \bigcup_{e \in E_2} I_e,$$

$$A_3 = \bigcup_{e \in E_3} I_e$$

and

$$A_4 = \bigcup_{e \in E_3} I_e.$$

First we get the trivial upper bound

$$\overline{\mu}(A_1) = \mu(B(x, r) \cap K) \le \mu(B(x, r)). \tag{4.2.33}$$

Use Case 8 of Lemma 4.2.25 to obtain

$$
\begin{aligned}
\overline{\mu}(A_2) \;&=\; \sum_{e \in E_2} \overline{\mu}(I_e) \\
&\leq\; \sum_{\substack{n \in N \\ d(n,x) \leq r \\ 0 < l(n) \leq r}} \;\sum_{\substack{s \leq l(n) \\ e \in E_{(n,s)}}} \overline{\mu}(I_e) \\
&\leq\; \sum_{\substack{n \in N \\ d(n,x) \leq r \\ 0 < l(n) \leq r}} 2^{12} D^{22} \mu(B(n, l(n))) \qquad\qquad (4.2.34)
\end{aligned}
$$

Next we use the map from Case 9 of Lemma 4.2.25 and doubling to bound $\mu(B(n, l(n))) \leq D^6 \mu(B(\rho(n), \mathbf{Sc}_G(\rho(n))2^{-4}))$, and $d(x, \rho(n)) \leq 2^5 r$. Also, the balls $B(\rho(n), \mathbf{Sc}_G(\rho(n))2^{-15}) \subset B(x, 2^5 r)$ are disjoint by (4.2.9). All in all, we can apply a union bound with (4.2.34) to conclude

$$
\begin{aligned}
\overline{\mu}(A_2) \;&\leq\; \sum_{\substack{n \in N \\ d(n,x) \leq r \\ 0 < l(n) \leq r}} 2^{12} D^{22} \mu(B(n, l(n))) \\
&\leq\; 2^{12} D^{40} \sum_{\substack{n \in N \\ d(n,x) \leq r \\ 0 < l(n) \leq r}} \mu(B(\rho(n), \mathbf{Sc}_G(\rho(n))2^{-15})) \\
&\leq\; 2^{12} D^{40} \sum_{\substack{g \in G \\ d(g,x) \leq 2^5 r}} \;\sum_{n \in \rho^{-1}(g)} \mu(B(\rho(n), \mathbf{Sc}_G(\rho(n))2^{-4})) \\
&\leq\; 2^{12} D^{60} \sum_{\substack{g \in G \\ d(g,x) \leq 2^5 r}} \mu(B(g, \mathbf{Sc}_G(g)2^{-4})) \\
&\leq\; 2^{12} D^{60} \mu(B(x, 2^5 r)) \leq 2^{12} D^{65} \mu(B(x, r)) \qquad (4.2.35)
\end{aligned}
$$

Next, we estimate $\overline{\mu}(A_3)$. Use Case 8 of Lemma 4.2.25 to obtain

$$\overline{\mu}(A_3) = \sum_{e \in E_3} \overline{\mu}(I_e)$$

$$\leq \sum_{\substack{n \in N \\ d(n,x) \leq r \\ r \leq l(n)}} \sum_{\substack{s \leq r \\ e \in E_{(n,s)}}} \overline{\mu}(I_e)$$

$$\leq 2^{12} D^{23} \sum_{\substack{n \in N \\ d(n,x) \leq r \\ r \leq \mathbf{Sc}_N(n)}} \mu(B(n, r/2)) \qquad (4.2.36)$$

Since $\mathbf{Sc}_N(n) \geq r$, the net-points n included in the sum are r-separated. Therefore, the balls $B(n, r/2)$ are disjoint. Also, $B(n, r/2) \subset B(x, 2r)$. We obtain

$$\overline{\mu}(A_3) \leq 2^{12} D^{23} \sum_{\substack{n \in N \\ d(n,x) \leq r \\ r \leq \mathbf{Sc}_N(n)}} \mu(B(n, r/2))$$

$$\leq 2^{12} D^{23} \mu(B(x, 2r)) \leq 2^{12} D^{24} \mu(B(x, r)) \qquad (4.2.37)$$

Finally, we estimate $\overline{\mu}(A_4)$. Consider some $v = (g, r) \in V_G$ and $e \in E_v$ such that $d(g, x) \leq r$. By (4.2.8) we have $\mathbf{Sc}(g)/2 \leq r$. Then by Case 7 $\overline{\mu}(I_e) \leq 2 D^7 \mu(B(g, \mathbf{Sc}(g))) \leq 2 D^{25} \mu(B(g, \mathbf{Sc}(g) 2^{-15}))$. For different v the balls $B(g, \mathbf{Sc}(g) 2^{-15})$ are disjoint by (4.2.9) and $B(g, \mathbf{Sc}(g) 2^{-15}) \subset B(x, 4r)$. Combine all these to see that

$$
\begin{aligned}
\overline{\mu}(A_4) \;&=\; \sum_{e \in E_3} \overline{\mu}(I_e) \\
&\leq\; 2D^{25}\mu(B(x,4r)) \leq 2D^{30}\mu(B(x,r)) \qquad\qquad (4.2.38)
\end{aligned}
$$

Combining the estimates (4.2.33), (4.2.35), (4.2.37), (4.2.38), with doubling for the different sets gives

$$
\overline{\mu}(\overline{B}(x,r)) \leq 2^{16}D^{65}\mu(B(x,r)). \qquad\qquad (4.2.39)
$$

- Combine estimates (4.2.32) and (4.2.39) for $x \in K, 0 < r < 1/2$ to give

$$
\frac{\overline{\mu}(\overline{B}(x,2r))}{\overline{\mu}(\overline{B}(x,r))} \;\leq\; 2^{20}D^{100}. \qquad\qquad (4.2.40)
$$

Case 2: $x \in e_x = ((a,s),(b,t)) \in E, \; 0 < r < s/4$**:** Denote $v = (a,s)$ and $w = (b,t)$. Let E_v, E_w be the sets of edges adjacent to either vertex. It is easy to conclude that

$$
\overline{B}(x,r) \subset \overline{B}(x,2r) \subset \bigcup_{e \in E_v \cup E_w} I_e. \qquad\qquad (4.2.41)
$$

For each $e \in E_v \cup E_w$ and any $h > 0$ we have $\mathrm{diam}(B(x,h) \cap e) \leq 2h$. Also, $|e| \geq s$ by (4.2.13). Further, we derive from (4.2.15) that in terms of densities

$$\mu_e \leq \frac{2\mu(B(a, 2^{10}s))}{|e|}\lambda \leq \frac{2\mu(B(a, 2^{10}s))}{s}\lambda.$$

Thus using $h = 2r$, the size bound Lemma 4.2.23 we get

$$
\begin{aligned}
\overline{\mu}(\overline{B}(x, 2r)) &= \sum_{e \in E_v \cup E_w} \mu(I_e \cap B(x, 2r)) \\
&\leq \frac{8r}{s}\mu(B(a, 2^{10}s)) \sum_{e \in E_v \cup E_w} 1 \\
&\leq \frac{2^{10}D^{35}r\mu(B(a, s))}{s}.
\end{aligned}
\tag{4.2.42}
$$

On the other hand for e_x we have $\mathrm{diam}(B(x, r) \cap e_x) \geq r$, and using $|e| \leq 2^7 s$ from Case 7 in Lemma 4.2.25 we see that in terms of densities

$$\mu_e \geq \frac{\mu(B(a, s))}{|e|}\lambda \geq \frac{\mu(B(a, s))}{2^7 s}\lambda.$$

Using this we obtain the lower bound

$$
\begin{aligned}
\overline{\mu}(\overline{B}(x, r)) &\geq \mu_e(e \cap B(x, r)) \\
&\geq \frac{r\mu(B(a, s))}{2^7 s}.
\end{aligned}
\tag{4.2.43}
$$

Combine estimates (4.2.43) and (4.2.42) to get the desired doubling bound

$$\frac{\overline{\mu}(\overline{B}(x, 2r))}{\overline{\mu}(\overline{B}(x, r))} \leq 2^{20}D^{35}.$$

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Case 3: $x \in e = ((a,s),(b,t)) \in E$, $s/4 < r < 2^{12}s$ **and** $r < 2^{-13}$: Let $v = (a,s)$.

By the estimate (4.2.43) and using that for $r2^{-12} < s/4$, we get

$$\overline{\mu}(\overline{B}(x,r)) \geq \overline{\mu}(\overline{B}(x,r2^{-14})) \geq \frac{r\mu(B(a,s))}{2^{30}r} \geq 2^{-30}\mu(B(a,r2^{-12})). \qquad (4.2.44)$$

By Case 3 of Lemma 4.2.25 there is a $n \in N$ such that $\overline{d}(v,n) \leq 2^6 s$. In particular, $\overline{d}(n,x) \leq \overline{d}(v,n) + |e| \leq 2^7 s + 2^6 s \leq 2^{10}s$. Apply the estimate (4.2.39) to see

$$\overline{\mu}(\overline{B}(x,2r)) \leq \overline{\mu}(\overline{B}(n,2^{11}r)) \leq 2^{15}D^{65}\mu(B(n,2^{11}r)). \qquad (4.2.45)$$

Using doubling with Cases 3 and 4 in Lemma 4.2.25 we get $\mu(B(n,2^{11}r)) \leq \mu(B(a,2^{12}r)) \leq D^{25}\mu(B(a,2^{-12}r))$. Finally, combining this with (4.2.44) we get the desired doubling bound

$$\frac{\overline{\mu}(\overline{B}(x,2r))}{\overline{\mu}(\overline{B}(x,r))} \leq 2^{45}D^{90}.$$

Case 4: $x \in e = ((a,s),(b,t)) \in E$, $2^{12}s < r$ **and** $r < 1/8$: Using Cases 3 and 7 of Lemma 4.2.25 we can find a n

$$\overline{d}(x,n) \leq 2^8 s \leq r2^{-4}.$$

In particular, we have

$$\overline{B}(n, r2^{-4}) \subset \overline{B}(x, r) \subset \overline{B}(x, 2r) \subset \overline{B}(n, 4r).$$

Thus using (4.2.40), we obtain

$$\frac{\overline{B}(x, 2r)}{\overline{B}(x, r)} \leq \frac{\overline{\mu}(\overline{B}(n, 4r))}{\overline{\mu}(\overline{B}(n, r2^{-4}))} \leq 2^{200} D^{500}.$$

These cases conclude the proof for doubling. Denote the maximum of the previous doubling constants $\overline{D} = 2^{200} D^{500}$. The cases above show that $(\overline{K}, \overline{d}, \overline{\mu})$ is $(\overline{D}, 2^{-13})$-doubling.

Connectivity: Next, we move to prove connectivity. We will show that every pair of points (x, y), which is sufficiently close, is $(\overline{C}, \frac{1}{2}, \overline{\epsilon})$-connected for appropriately chosen $\overline{\epsilon}$ and $\overline{C}$. Again we have cases depending on the positions of x and y. The most complicated case is when both $x, y \in K$.

We always denote $r = \overline{d}(x, y)$. For each pair of points take an arbitrary open set E such that $\overline{\mu}(E \cap \overline{B}(x, Cr)) \leq \eta \overline{\mu}(\overline{B}(x, Cr))$, where η, C are the relevant connectivity parameters in each case. Also, it is enough to verify Definition 3.1.2 for open sets, because of the regularity of the measure. Here, we have already shown that $\overline{\mu}$ is a bounded measure on a compact metric space, and thus $\overline{\mu}$ is a Radon measure, and both inner and outer regular (see [104]). Note, by doubling and Lemma 4.2.22, we don't need to be so concerned about the parameter C, which represents the length of the curve fragments, being the same in each case. At the end $\overline{C}$ will become the maximum from the different C's in each case, and we will adjust $\overline{\epsilon}$ to account for this.

For technical reasons, some cases involve seemingly stronger statements than necessary. Again, we will be somewhat liberal in our estimates. For example, we will use $\overline{D}$ in places where D would suffice. Together these cases show that each $(x, y) \in \overline{X} \times \overline{X}$ with $\overline{d}(x, y) \le 2^{-200}C^{-1}$ is

$$\left(2^{35}C, 2^{-14}, \epsilon(2\overline{D})^{-2000-2\log_2(C)}\right)\text{-connected.}$$

Case 1: Let $\delta > 0$ be arbitrary and assume $(x, y) \in T \times T$ with $\overline{d}(x, y) < 2^{-6}$, and $x \in e = ((a, s), (b, t)) \in E$. If $\overline{d}(x, y) < s2^{-4}$, then the pair (x, y) is $(1, \delta, \delta(2\overline{D})^{-10+\log_2(\delta)})$-connected.

Let $v = (a, s)$ and $w = (b, t)$. By Case 2 in 4.2.25 and (4.2.13) we know that $x, y \in \cup_{e \in E_v \cup E_w} I_e = A$. Lemma 4.2.23 gives that A is a connected subgraph of T with at most $4D^{13}$ edges. The metric restricted to A agrees with the path metric d_T by similar arguments to those used in Estimate 2 in Lemma 4.2.25. The path connecting any pair of points passes through at most three intervals. The result follows now easily from Lemmas 4.2.21 and 4.2.20.

Case 2: Every pair $(x, y) \in K \times K$ with $d(x, y) \le 2^{-200}C^{-1}$ is

$$\left(2^{20}C, 2^{-30}, \epsilon\left(2\overline{D}\right)^{-1000-\log_2(C)}\right)\text{-connected.}$$

Let $E \subset \overline{B}(x, 2^{20}Cd(x, y))$ be a set such that

$$\overline{\mu}(E) \le \epsilon\left(2\overline{D}\right)^{-1000-\log_2(C)} \overline{\mu}(\overline{B}(x, 2^{20}Cd(x, y))). \qquad (4.2.46)$$

Our goal will be to first construct a curve fragment in A and then to replace portions of it with a curve in T. This latter step is only possible, if the portions of the curve fragment in A doesn't pass too close to certain "bad" gap points. First, we define bad bridge points.

Consider the collection $\mathbf{B}$ of "bad" bridge points $b \in V$ with $\mathbf{Sc}(b) < 2^{-50}$ such that

$$\overline{\mu}(\overline{B}(b, 2^{10}\mathbf{Sc}(b)) \cap E) \geq \left(2\overline{D}\right)^{-\log_2(C)-500} \overline{\mu}\left(\overline{B}\left(b, 2^{10}\mathbf{Sc}(b)\right)\right). \qquad (4.2.47)$$

Define

$$\mathcal{B} = \bigcup_{b \in \mathbf{B}} B(\mathbf{Loc}(b), \mathbf{Sc}(b))$$

and its approximant in $\overline{K}$

$$\overline{\mathcal{B}} = \bigcup_{b \in \mathbf{B}, e \in E_b} I_e.$$

We will seek to estimate the volume of $\mathcal{B}$ from above. Each of the edges I_e can appear at most twice in the union defining $\overline{\mathcal{B}}$. Use Case 7 of Lemma 4.2.25 to see that

$$\begin{aligned}
\mu(\mathcal{B}) &\leq \sum_{b \in \mathbf{B}} \mu(B(\mathbf{Loc}(b), \mathbf{Sc}(b))) \\
&\leq \sum_{b \in \mathbf{B}, e \in E_b} \overline{\mu}(I_e) \\
&\leq 2\overline{\mu}(\overline{\mathcal{B}}).
\end{aligned} \qquad (4.2.48)$$

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We can use the Vitali covering Lemma [55, 135] to choose a sub-collection $\mathbf{B}'$ of $b \in \mathbf{B}$ such that

$$\overline{\mathcal{B}} \subset \bigcup_{b \in \mathbf{B}'} \overline{B}(b, 2^{13} \mathbf{Sc}(b))$$

and the balls $\overline{B}(b, 2^{10} \mathbf{Sc}(b))$ for $b \in \mathbf{B}'$ are disjoint. Apply doubling of $\overline{\mu}$, which we have already found, and the definition (4.2.47) to get the following.

$$
\begin{aligned}
\overline{\mu}(E) \;&\geq\; \sum_{b \in \mathbf{B}'} \overline{\mu}(\overline{B}(b, 2^{10} \mathbf{Sc}(b)) \cap E) \\
&\geq\; (2\overline{D})^{-\log_2(C) - 500} \sum_{b \in \mathbf{B}'} \overline{\mu}\left(\overline{B}(b, 2^{10} \mathbf{Sc}(b))\right) \\
&\geq\; (2\overline{D})^{-\log_2(C) - 503} \sum_{(k,m) \in \mathbf{B}'} \overline{\mu}\left(\overline{B}(b, 2^{13} \mathbf{Sc}(b))\right) \\
&\geq\; (2\overline{D})^{-\log_2(C) - 503} \overline{\mu}(\overline{\mathcal{B}}) \qquad\qquad\qquad (4.2.49)
\end{aligned}
$$

Thus, using (4.2.46) with doubling we see

$$\overline{\mu}(\overline{\mathcal{B}}) \leq \epsilon \left(2\overline{D}\right)^{-400} \overline{\mu}\left(\overline{B}(x, Cd(x, y))\right). \qquad (4.2.50)$$

Thus with the estimates (4.2.46), (4.2.48) and (4.2.33), we see that

$$\mu(\mathcal{B} \cup (K \cap E)) \leq 2\overline{\mu}(\overline{\mathcal{B}}) + \overline{\mu}(E) \leq \epsilon\mu\left(B(x, Cd(x, y))\right). \qquad (4.2.51)$$

By using the connectivity condition, we can find a 1-Lipschitz curve fragment $\gamma \colon S \to A$, which is parametrized by length (see Lemma 3.2.7) with $\gamma(0) = x$, $\gamma(\text{len}(\gamma)) = y$, $|\text{Undef}(\gamma)| < 2^{-60} d(x, y)$, $\text{len}(\gamma) \leq Cd(x, y)$ and $\gamma^{-1}(\mathcal{B} \cup (K \cap E))) \subset \{0, \max(S)\}$.

The image of the curve fragment γ may not be contained in K. Thus, to reach our desired conclusion we will define another curve fragment $\overline{\gamma}$ in $\overline{K}$ by replacing the portions in A/K with curves in $\overline{K}$. This is done in two steps. First we discretize the portions of the fragment in $A \setminus K$ and obtain a "discrete path fragment" through gap points g. Intuitively, the sub-segments of the curve fragment associated to gap points g will be related to the corresponding bridge point $b \in V$. In the construction, we can guarantee that $b \notin \mathbf{B}$, and thus we will be able to connect sufficiently many consecutive bridge points to each other and go from a discrete path to a continuous curve fragment. This latter part of the argument is called extension. Finally, the curve fragment we construct won't satisfy the desired Lipschitz-bound so we will need to dilate the domain by a certain factor and thus complete the proof.

1. **Discretization:** First, we cover the set $\gamma^{-1}(A \setminus K) = O$ by intervals as follows. Recall the Whitney covering property (4.2.7), the length estimate, and the fact that $\gamma(O) \cap \mathcal{B} = \emptyset$. Using these, choose for each $z \in O$ a g_z with the property $d(g_z, \gamma(z)) \leq \mathbf{Sc}_G(g_z)2^{-10}$. It is easy to see that $b_z = (g_z, \mathbf{Sc}_G(g_z)) \notin \mathbf{B}$. Define the interval

$$I_z = (z - \mathbf{Sc}_G(g_z)2^{-10}, z + \mathbf{Sc}_G(g_z)2^{-10}) \tag{4.2.52}$$

and the smaller interval $J_z = (z - \mathbf{Sc}_G(g_z)2^{-15}, z + \mathbf{Sc}_G(g_z)2^{-15})$ and choose centers $z \in \overline{Z} \subset O$ such that

$$O \subset \bigcup_{z \in \overline{Z}} I_z,$$

and J_z are pairwise disjoint for $z \in \overline{Z}$. This is possible by the Vitali covering theorem [55, 135]. Because γ is 1-Lipschitz, we obtain $I_z \subset [0, \max(S)] \setminus \gamma^{-1}(K)$ for all $z \in \overline{Z}$.

We will define a "discrete curve fragment" as follows. Consider the compact set $\overline{S}_0 = \gamma^{-1}(K) \cup \overline{Z}$ and the curve fragment $\overline{\gamma}_0 \colon \overline{S}_0 \to \overline{K}$ by setting $\overline{\gamma}(t) = \gamma(t)$ for $t \in S$ and $\overline{\gamma}_0(z) = b_z$ for $z \in \overline{Z}$. The crucial properties we need are:

(a) $\max(\overline{S}_0) = \max(\overline{S}) \le C d(x, y)$,

(b) $\mathrm{LIP}\,(\overline{\gamma}_0) \le 2^{20}$.

The first estimate is immediate from the definition. Since the curve fragment is 1-Lipschitz on $\gamma^{-1}(K)$ (this involves Estimate 1 from Lemma 4.2.25), it is sufficient to check the Lipschitz-bound for pairs of points $z, w \in \overline{Z}$ and $z \in \overline{Z}$ and $w \in \gamma^{-1}(K)$.

Consider the case where both $z, w \in \overline{Z}$. Then by the triangle inequality

$$d(g_z, g_w) \le d(g_z, \gamma(z)) + d(\gamma(z), \gamma(w)) + d(g_w, \gamma(w)).$$

Further by the Lipschitz bound $d(\gamma(z), \gamma(w)) \le |z - w|$, and by the choice of g_z and g_w , for $t = z, w$ we get $d(g_t, \gamma(t)) \le \mathbf{Sc}_G(g_t) 2^{-10}$. Also, $J_z \cap J_w = \emptyset$, so $\mathbf{Sc}_G(g_z) 2^{-15} + \mathbf{Sc}_G(g_w) 2^{-15} \le |z - w|$. All these combined give $d(g_z, g_w) \le 2^7 |z - w|$.

Use Estimate 6 from Lemma 4.2.25 to give the Lipschitz-bound.

$$\bar{d}(\bar{\gamma}_0(z), \bar{\gamma}_0(w)) \;=\; \bar{d}(b_z, b_w)$$
$$\leq\; d(g_z, g_w) + 2^4(\mathbf{Sc}_G(g_z) + \mathbf{Sc}_G(g_w))$$
$$\leq\; 2^{20}|z - w|.$$

Consider next the case $z \in \overline{Z}$ and $w \in \gamma^{-1}(K)$. Then by the triangle inequality

$$d(g_z, \gamma(w)) \leq d(g_z, \gamma(z)) + d(\gamma(z), \gamma(w)).$$

Further by the Lipschitz bound $d(\gamma(z), \gamma(w)) \leq |z - w|$, and by the choice of g_z and Estimate (4.2.8) we get $d(g_z, \gamma(z)) \leq \mathbf{Sc}_G(g_z)2^{-10} \leq d(g_z, K)2^{-9} \leq d(g_z, \gamma(w))2^{-9}$. Thus

$$d(g_z, \gamma(w)) \leq 2|z - w|.$$

Also, $\mathbf{Sc}_G(g_z) \leq 2d(g_z, K) \leq 4|z - w|$. Use Estimate 4 from Lemma 4.2.25 for $b_z = (g_z, \mathbf{Sc}_G(g_z))$ to give the Lipschitz-bound

$$\bar{d}(\bar{\gamma}_0(z), \bar{\gamma}_0(w)) = \bar{d}(b_z, \gamma(w)) \leq d(g_z, \gamma(w)) + 2^8 \mathbf{Sc}_G(g_z) \leq 2^{20}|z - w|.$$

2. **Extension:** The desired curve fragment $\overline{\gamma}$ will result from expanding $\overline{\gamma}_0$ for certain "consecutive" pairs of $z, w \in \overline{Z}$. We call $(z, w) \in \overline{S}_0 \times \overline{S}_0$ *consecutive* if $z < w$ and $(z, w) \cap \overline{S}_0 = \emptyset$. For each consecutive pair $(z, w) \in \overline{Z} \times \overline{Z}$ consider the interval $I_{z,w} = [z, w]$. Let $\mathcal{I} = \{I_{z,w}\}$ be the collection of all such intervals. Then, let $\mathcal{G}$ be the collection of "good" intervals $I_{z,w}$ such that $z, w \in \overline{Z}$ and

$$d(g_z, g_w) \leq 2^{-3}(\mathbf{Sc}_G(g_z) + \mathbf{Sc}_G(g_w)), \tag{4.2.53}$$

and finally let $\mathcal{F} = \mathcal{I} \setminus \mathcal{G}$.

Define $\overline{S} = \overline{S}_0 \cup \bigcup_{I \in \mathcal{G}} I$, and the curve fragment $\overline{\gamma} \colon \overline{S} \to \overline{K}$ by $\overline{\gamma}(t) = \overline{\gamma}_0(t)$ for $t \in \overline{S}_0$, and $\overline{\gamma}$ is a linear parametrization of the edge $I_{(b_z, b_w)} \subset T$ for $I_{z,w} \in \mathcal{G}$. This edge exists by the following argument.

Firstly, $\mathbf{Sc}(g_z)/2 < d(g_z, K) \leq d(g_z, g_w) + d(g_w, K)$. Thus, $\mathbf{Sc}(g_z)3/8 < \frac{9}{8}\mathbf{Sc}(g_w)$ follows from estimates (4.2.53) and (4.2.8). By symmetry, we obtain $|\mathbf{Sc}(g_z)/\mathbf{Sc}(g_w)| < 4$, and thus (4.2.12) must hold. On the other hand, ensuring (4.2.11) is a triviality.

3. **Estimates:** We will show that this curve fragment satisfies the following.

   (a) LIP $(\overline{\gamma}) \leq 2^{20}$

   (b) $\operatorname{len}(\overline{\gamma}) \leq 2^{20} Cd(x, y)$

   (c) $|\mathrm{Undef}(\overline{\gamma})| < 2^{-31} d(x, y)$

(d) $|\bar{\gamma}^{-1}(E)| < 2^{-31}d(x,y)$

From these the desired curve fragment satisfying the conditions of Definition 3.1.2 can be obtained by restricting $\bar{\gamma}$ to a large compact set in the complement of $\bar{\gamma}^{-1}(E)$ and by dilating the domain by a factor of 2^{20}. Thus, we are left with proving these four estimates.

The Lipschitz estimate is trivial, since the domain is expanded by linear parametrizations. Here we also use Estimate 2 in Lemma 4.2.25. Similarly, the length estimate follows because $\mathrm{len}(\bar{\gamma}) = \mathrm{len}(\bar{\gamma}_0) \leq 2^{20}\max(\bar{S}_0) \leq 2^{20}\mathrm{len}(\gamma)$.

Next, estimate $\mathrm{Undef}(\bar{\gamma})$. It is easy to see $\mathrm{Undef}(\bar{\gamma}) \subset \mathrm{Undef}(\gamma) \cup \bigcup_{I \in \mathcal{F}} I$. Let $I \in \mathcal{F}$ be an arbitrary interval. Consider the interval $C_I = (z,w) \setminus (I_z \cup I_w)$. Clearly $C_I \subset \mathrm{Undef}(\gamma)$, because $C_I \cap K = \emptyset$. We will show in the following paragraph that $|I| \leq 2|C_I|$. Assuming this for now, we will complete the estimate. Since the intervals $I \in \mathcal{F}$ are disjoint, so are $C_I \subset I$. Also,

$$
\begin{aligned}
|\mathrm{Undef}(\bar{\gamma})| &\leq |\mathrm{Undef}(\gamma)| + \sum_{I \in \mathcal{F}} |I| \\
&\leq |\mathrm{Undef}(\gamma)| + \sum_{I \in \mathcal{F}} 2|C_I| \\
&\leq 3|\mathrm{Undef}(\gamma)| \leq 2^{-31}d(x,y).
\end{aligned}
$$

For each $I = I_{z,w} \in \mathcal{F}$ we have $d(g_z, g_w) > 2^{-3}(\mathbf{Sc}_G(g_z) + \mathbf{Sc}_G(g_w))$ and $(z,w) \cap \bar{S} = \emptyset$. Thus, $|z - w| \geq d(\gamma(z), \gamma(w)) \geq 2^{-4}(\mathbf{Sc}_G(g_z) + \mathbf{Sc}_G(g_w))$. Also, from definition (4.2.52) and the previous, we get

$$\begin{aligned}
|C_I| &= |(z,w) \setminus (I_z \cup I_w)| \\
&\geq 2^{-4}(\mathbf{Sc}_G(g_z) + \mathbf{Sc}_G(g_w)) - 2^{-10}(\mathbf{Sc}_G(g_z) + \mathbf{Sc}_G(g_w)) \\
&\geq \mathbf{Sc}_G(g_w))2^{-5}(\mathbf{Sc}_G(g_z) + \mathbf{Sc}_G(g_w)).
\end{aligned}$$

Finally, from (4.2.52),

$$|I| \leq |C_I| + 2^{-10}\mathbf{Sc}_G(g_z) + 2^{-10}\mathbf{Sc}_G(g_w) \leq 2|C_I|.$$

The remaining estimate concerns $|\overline{\gamma}^{-1}(E)|$. For each $I = (z,w) \in \mathcal{G}$ denote by $e_I = (b_z, b_w)$ the edge in $T \subset \overline{K}$ corresponding to it. Note that

$$|e_I| \leq 2^{20}|z - w| = 2^{20}|I| \tag{4.2.54}$$

by the Lipschitz estimate of $\overline{\gamma}$. Assume without loss of generality that $\mathbf{Sc}(b_w) \leq \mathbf{Sc}(b_z)$. Since $b_z, b_w \notin \mathcal{B}$, we have by Estimate (4.2.47)

$$\overline{\mu}(\overline{B}(b_z, 2^{10}\mathbf{Sc}(b_z)) \cap E) < \left(2\overline{D}\right)^{-\log_2(C)-500} \overline{\mu}\left(\overline{B}\left(b_z, 2^{10}\mathbf{Sc}(b_z)\right)\right). \tag{4.2.55}$$

Thus, by doubling estimates similar to Lemma 4.2.21, the estimate $\mathbf{Sc}(b_w) \leq \mathbf{Sc}(b_z)$ and estimate (4.2.13) we have

$$\overline{\mu}(e_I \cap E) \leq \left(2\overline{D}\right)^{-\log_2(C)-400} \overline{\mu}(e_I). \tag{4.2.56}$$

Since $\overline{\gamma}$ is a linear parametrization of e_I with the interval I, we have also

$$|\overline{\gamma}^{-1}(E) \cap I| \leq \left(2\overline{D}\right)^{-\log_2(C)-300} |I|. \tag{4.2.57}$$

Thus, we get

$$
\begin{aligned}
|\overline{\gamma}^{-1}(E)| &\leq \sum_{I \in \mathcal{G}} |\overline{\gamma}^{-1}(E) \cap I| \\
&\leq \sum_{I \in \mathcal{G}} \left(2\overline{D}\right)^{-\log_2(C)-300} |I| \\
&\leq \left(2\overline{D}\right)^{-\log_2(C)-300} Cd(x,y) \leq 2^{-31} d(x,y).
\end{aligned}
$$

Case 3. **For any $\delta > 0$ and every $x \in e = ((a,s),(b,t)) \in T$ with $s <$ 2^{-6} there is a $y \in K$ such that $\overline{d}(x,y) \leq 2^8 s$ and such that (x,y) is $(2^9, \delta, \delta(2\overline{D})^{2\log_2(\delta)-40})$-connected.** Let $v = (a,s), w = (b,t)$. By Case 3 in Lemma 4.2.25 there is a $n \in N$ such that $\overline{d}(n,v) \leq 2^6 s$. Further, $((n,s),v) \in E$. Set $y = n$.

Next, take $k = \lceil 20 - \log_2(\delta) \rceil$. Define the points $p_1 = x, p_2 = v, p_{k+1} = n = y$ and $p_i = (n, s2^{3-i})$ for $i = 3, \ldots, k$. By Case 5 or Lemma 4.2.25 we have

$$\overline{d}(p_k, y) \leq 2^8 \mathbf{Sc}(p_k) \leq 2^{\log_2(\delta)-5} s \leq \delta/4 \overline{d}(x,y).$$

Each pair (p_i, p_{i+1}) are edges in T by condition (4.2.11) and (4.2.12). Thus, by Lemma 4.2.21, each pair (p_i, p_{i+1}), for $1 \leq i \leq k-1$, is $(1, \delta 2^{-11}, \delta(2\overline{D})^{-14})$-connected. Also,

$$\sum_{i=1}^{k-1} \overline{d}(p_i, p_{i+1}) \leq 2^8 s,$$

and $\overline{d}(p_1, p_k) \geq s/2$. Further by Lemma 4.2.20 (using $L = 2^9$) and distance estimates from Lemma 4.2.25, the pair (p_1, p_k) is $(2^9, \delta/2, \delta(2\overline{D})^{-30-\log_2(\delta)-\log_2(k)})$-connected. We observe that $\overline{d}(p_k, p_{k+1}) \leq \delta/4\overline{d}(x, y)$. Thus, any curve fragment connecting p_1 to p_k can be enlarged by a small gap to one connecting p_1 to p_{k+1}. This changes the length slightly, so by repeating an argument from Lemma 4.2.22, we see that the pair $(x, y) = (p_1, p_{k+1})$ is $(2^9, \delta, \delta(2\overline{D})^{2\log_2(\delta)-40})$-connected.

Case 4. Each pair $(x, y) \in T \times T$ with $0 < \overline{d}(x, y) \leq 2^{-100}C^{-1}$ is

$$(2^{35}C, 2^{-14}, \epsilon(2\overline{D})^{-2000-2\log_2(C)})\text{-connected.}$$

Let $x \in e_x = (v_x, w_x)$ and $y \in e_y = (v_y, v_w)$. If $\overline{d}(x, y) < \mathbf{Sc}(v_x)2^{-4}$ or $\overline{d}(x, y) < \mathbf{Sc}(v_y)2^{-4}$, the result follows from Case 1. Thus, assume that

$$\overline{d}(x, y) \geq \max(\mathbf{Sc}(v_x)2^{-4}, \mathbf{Sc}(v_y)2^{-4}).$$

Let n_x, n_y be as in Case 3. In other words $\overline{d}(n_x, x) \leq 2^8\mathbf{Sc}(v_x) \leq 2^{12}\overline{d}(x, y)$, $\overline{d}(n_y, y) \leq 2^8\mathbf{Sc}(v_y) \leq 2^{12}\overline{d}(x, y)$ and (n_x, x) and (n_y, y) are $(2^9, 2^{-30}, (2\overline{D})^{-200})$-connected. Now, consider the discrete path $p_1 = x, p_2 = n_x, p_3 = n_y, p_4 = y$. Note, that (p_2, p_3) is $(2^{20}C, 2^{-30}, \epsilon(2\overline{D})^{-1000-\log_2(C)})$-connected by Case 2. We have

$$\sum_{i=1}^{3} \overline{d}(p_i, p_{i+1}) \leq 2^{15}\overline{d}(x, y).$$

Thus from Lemma 4.2.20, we get that (x, y) is $(2^{35}C, 2^{-14}, \epsilon(2\overline{D})^{-2000-2\log_2(C)})$-connected.

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Case 5. For each $x \in T$ and $y \in K$ the pair (x, y) with $d(x, y) \leq 2^{-100}C^{-1}$ is

$$(2^{35}C, 2^{-14}, \epsilon(2\overline{D})^{-2000-2\log_2(C)})\text{-\textbf{connected}}.$$

Define n_x as before. Then we can consider the discrete path given by $p_1 = x, p_2 = n_x$ and $p_3 = y$. The result follows analogously to Case 4 from Lemma 4.2.20.

Concluding remarks: We have established $(\overline{D}, 2^{-5})$-doubling and

$$\left(2^{35}C, 2^{-14}, \epsilon(2\overline{D})^{-2000-2\log_2(C)}, 2^{-100}C^{-1}\right)\text{-connectivity.}$$

By Theorem 3.3.17 we obtain that the space $(\overline{X}, \overline{d}, \overline{\mu})$ is a PI-space with appropriate parameters. The constants in this theorem almost certainly could be substantially improved, but that is not relevant for us.

$\square$

In the final section we give a result for tangents of RNP-Lipschitz differentiability spaces which can be seen as a corollary of the previous result. However, the alternative proof is simpler.

4.3 Tangents of RNP-Lipschitz differentiability spaces

We give a shorter proof that tangents of RNP-Lipschitz differentiability spaces are almost everywhere PI-spaces. First a result on stability of the connectivity

criterion.

Theorem 4.3.1. *If* $(X_i, d_i, \mu_i, x_i) \to (X, d, \mu, x)$ *is a convergent sequence of proper pointed metric measure spaces in the measured Gromov-Hausdorff sense, and each X_i is (C, δ, ϵ)-connected, then also the limit space (X, d, μ) is (C, δ, ϵ)-connected.*

Proof: It is sufficient to assume $\delta < 1$, as otherwise the claim is trivial. Consider the spaces X_i, X embedded isometrically in some super-space (Z, d), such that the induced measures converge weakly and X_i converge to X in the Hausdorff sense (see for similar argument [84]). Let $x, y \in X$ be arbitrary with $d(x, y) = r$ and $E \subset B(x, Cr)$ be a set such that

$$\mu(E \cap B(x, Cr)) < \epsilon\mu(B(x, Cr)).$$

There is a $0 < \epsilon' < \epsilon$, such that

$$\mu(E \cap B(x, Cr)) < \epsilon\mu(B(x, Cr)).$$

By regularity, no generality is lost by assuming that E is open. Define a function on X for each (fixed) $\eta > 0$ by $\rho_\eta(z) = \max\left(\frac{d(z, E^c)}{\eta}, 1\right)$.

Extend the function to Z in such a way that it is compactly supported, Lipschitz and bounded by 1, and denote the extension by the same symbol. By weak convergence

$$\int_Z \rho_\eta d\mu_i \to \int_Z \rho_\eta d\mu \leq \mu(E). \tag{4.3.2}$$

Thus, the open sets $E_i = \left\{\rho_\eta > \frac{\epsilon'}{\epsilon}\right\} \cap X_i$ satisfy by (4.3.2)

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$$\limsup_{n\to\infty} \mu_n(E_n) \leq \lim_{n\to\infty} \frac{\epsilon}{\epsilon'} \int_{X_n} \rho_\eta \, d\mu_i$$
$$\leq \frac{\epsilon}{\epsilon'} \mu(E) < \epsilon \mu(B(x, Cr)). \qquad (4.3.3)$$

Further, choose sequences $x_i, y_i \in X_i$ such that $x_i \to x$ and $y_i \to y$. Our balls are assumed to be open, so from lower semi-continuity of the volume for open sets we see

$$\liminf_{n\to\infty} B(x_i, Cd(x_i, y_i)) \geq \mu(B(x, Cr)).$$

Finally, combining this with estimate (4.3.3) gives an N large enough such that for all $i > N$ it holds

$$\mu_i(E_i) < \epsilon B(x_i, Cd(x_i, y_i)).$$

We can now choose for all $i > N$ a 1-Lipschitz maps $\gamma_i \colon K_i \to X_i \subset Z$ defined on a compact subset $K_i \subset [0, 2Cd(x, y)]$, and satisfying

- $0 = \min(K_i)$ and $\max(K_i) \leq Cd(x_i, y_i) \leq 2Cd(x, y)$,

-
$$|[0, \max(K_i)] \setminus K_i| \leq \delta d(x_i, y_i), \text{and}$$

-
$$\rho_\eta(\gamma_i(t)) \leq \frac{\epsilon'}{\epsilon} < 1$$

for every $t \in K \setminus \{0, \max(K_i)\}$.

By an Arzela-Ascoli argument we can choose a subsequence i_k, a compact set

K such that $d_H(K_{i_k}, K) \to 0$, and a map $\gamma\colon K \to X \subset Z$ which is a limit of γ_{i_k}. Also, for any $t_k \in K_{i_k}$ such that $t_k \to t$ we have $t \in K$ and $\gamma_{i_k}(t_k) \to \gamma(t)$. From this it is easy to see $\gamma(0) = x$. Further, as $\lim_{k\to\infty} \max(K_{i_k}) = \max(K)$, we have $\gamma(\max(K)) = y$. Finally, using the continuity of ρ_η on Z we get $\rho_\eta(\gamma(t)) \leq \frac{\epsilon'}{\epsilon} < 1$ for any $t \in K \setminus \{0, \max(K)\}$. This means that $\gamma(t) \notin \{\rho_\eta > \frac{\epsilon'}{\epsilon}\}$ for the same range of t.

By upper semi-continuity with respect to Hausdorff convergence of the volume for compact sets

$$\limsup_{k\to\infty} |K_{i_k}| \leq |K|,$$

and further

$$|[0, \max(K)] \setminus K| = \max(K) - |K| \leq \lim\inf_{k\to\infty} \max K_{i_k} - |K_{i_k}| \leq \delta d(x, y).$$

This shows $|\mathrm{Undef}(\gamma)| \leq \delta d(x, y)$.

Let us now allow $\eta > 0$ to vary. For each η we can do the previous construction and define γ_η to satisfy the above conditions. Taking a subsequential limit with $\eta \to 0$, we get a 1-Lipschitz curve fragment γ with $|\mathrm{Undef}(\gamma)| \leq \delta d(x, y)$ and $\mathrm{len}(\gamma) \leq C(d, y)$. Further, for any $\eta > 0$ and any $t \in K_i \setminus \{\min(K_i), \max(K_i)\}$ we have $\rho_\eta(\gamma_\eta(t)) \leq \frac{\epsilon'}{\epsilon} < 1$. Thus, from the definition of ρ_η, we get

$$d(\gamma_\eta(t), E^c) \leq \eta \frac{\epsilon'}{\epsilon}.$$

Letting $\eta \to 0$ we get for the limiting curve $d(\gamma(t), E^c) \leq 0$ for $t \in K \notin$

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$\{\min(K), \max(K)\}$, and thus $\gamma(t)$, when defined and excluding end-points, is not in E.

$\square$

For the purposes of taking a tangent we need a slightly more general version of the previous theorem.

Theorem 4.3.4. *If $(X_i, d_{,i}, \mu_i, x_i) \to (X, d, \mu, p)$ is a convergent sequence of proper pointed metric measure and $S_i \subset X_i$ is a sequence of subsets such that X is $(C, \delta, \epsilon, r_i)$-connected along S_i and S_i is ϵ_i-dense in $B(x_i, R_i)$ where $\lim_{i \to \infty} R_i = \infty$ and $\lim_{i \to \infty} \epsilon_i = 0$, $\lim_{i \to \infty} r_i = \infty$, then the limit space (X, d, μ) is $(C, 2\delta, \frac{\epsilon}{2})$-connected.*

Proof: The proof proceeds exactly as before, and allows for constructing curve fragments γ_i in X that connect pairs of points in S_i, and taking their limits. The additional point we make, is that for any $x, y \in X$, there is a sequence of points $x_i, y_i \in S_i$ that converge to it.

$\square$

The following corollary would be a consequence of Theorem 1.2.62 but we provide an alternative and much simpler proof for it.

Corollary 4.3.5. *Let (X, d, μ) be an RNP-differentiability space. Then X can be covered by positive measure subsets V_i, such that each V_i is metric doubling, when equipped with its restricted distance, and for μ-a.e. $x \in V_i$ each space $M \in T_x(V_i)$ admits a PI-inequality of type $(1, p)$ for all p sufficiently large.*

Proof: By [12] we have a decomposition into sets $K_i^j \subset V_i$ with $X = \bigcup V_i \cup N = \bigcup K_i^j \cup N'$, such that each V_i and $K_i^j \subset V_i$ satisfies the following,

- X is uniformly (D_i, r_i)-doubling along V_i.

- X is uniformly $(C_i, 2^{-30}, \epsilon_i, r_i)$-connected along V_i.

- V_i is a uniform $(\frac{1}{2}, r_j^i)$-density set in X along K_i^j.

Blowing up V_i with its restricted measure and distance, at a point of K_i^j, gives the desired result by Lemma 4.3.4.

$\square$

Remark: The previous proof indicates the problem of applying the iterative procedure from Theorem 1.2.38 to conclude Poincaré inequalities on a RNP differentiability space. The constructed paths don't need to lie inside V_i, but instead only close by, and the closeness dictated by the density of the set and doubling. Since the curves may leave V_i, we can not guarantee the ability to refill their gaps.

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