Decomposability Bundles and Approximations of Lipschitz functions

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Chapter 1

Intro

These are lecture notes written for a four lecture mini course given at Peking University September 8th-13th 2025. They closely mirror the actual lectures, even though, as always, the presented word is rarely the same as the written word. In particular, the lectures contained some explanation that wasn't written here, and these notes are a bit more detailed and contain some results that I decided to skip in the lectures. In the end, I also didn't follow exactly the planned lecture division. Feel free to contact me about questions regarding these lecture notes.

There are a few references at the end of the notes. Unfortunately, it is difficult to give a reference for every stated fact, since many of these statements are not presented this way. Indeed, these notes introduce some techniques that are not published in this form anywhere, and I have taken some short cuts and simplifications for the sake of presentation.

In hindsight, I think Lectures 1 and 2 may be easier to follow than 3 and 4, and to fully grasp the ideas, one needs some background in geometric measure theory - and a whole lot of real analysis. Even just the simpler ideas in the first 2 sections are tremendously deep and useful. One could thus focus on them at first reading, and take the final sections mostly to be a source of inspiration.

Chapter 2

Lecture 1: Energies and Absolute continuity

2.1 The holy trinity

A key object in traditional potential analysis and complex analysis is the Dirichlet energy

$$\int_\Omega |\nabla u|^2 d\mathcal{L}.$$

It is associated with the heat semi-group P_t, the infinitesimal generator $-\Delta$, and Brownian motion. The study of these objects has been always closely intertwined with complex analysis. One reason for this is that the real and imaginary parts of a complex analytic function are harmonic, i.e. satisfy $\Delta u = 0$. Harmonic functions are local minimzers for the energy and correspond to stationary heat distributions.

If we step away from the complex plane, we consider general metric spaces (X, d) equipped with measures μ, and often need to consider exponents $p \neq 2$ above. For example, while the Dirichlet energy is conformally invariant in two dimensions, in $\mathbb{R}^n$ its place is taken by the $n-$energy

$$\int_\Omega |\nabla u|^n d\mathcal{L}.$$

In general metric spaces (X, d), to define a quasisymmetrically invariant energy, one needs one of much more involved construction, that is either based on upper gradients or re-scaled discrete energies. These energies yield invariants of group actions, natural quasisymmetric invariants, and their study reveals deep properties of the underlying space – for example, whether the Ahlfors regular conformal dimension is attained.

That there are two very different approaches to extending energies is quite fascinating. Indeed, it gives birth two two fields that have existed for quite some time: Analysis on Metric spaces for 30 years and Analysis on Fractals/the study of general Dirichlet forms for nearly 60 years. These fields overlap, but have developed largely independently. This independence leads to some curious historical coincidences. In the 1980s Bouleau-Hirsh posed the energy image density conjecture for Dirichlet forms. This conjecture was well-known in that community but remained open to this day. Meanwhile in the 1990s Cheeger developed a deep differentiation theory in metric measure spaces, and this lead to what is called Cheeger's conjecture. This conjecture, strikingly, was resolved by the works of de Philippis and Rinlder, Alberti-Marchese, Bate, and Alberti-Csörnyei-Preiss. It turns out, that this solution holds the keys to the resolution of the Bouleau-Hirsch conjecture, with methods from an entirely indpendent approach, and with a hint of PDE.

The key issue that arises in this theory and conjecture, is that the energies used are quite abstract, and hard or impossible to explicitly compute. In many cases, we can not even give explicit formulas for harmonic functions or any functions with finite energy. This is the setting where the techniques of these lectures come into play. A key lesson for us has been, that the Lipschitz analysis on $\mathbb{R}^n$ can be used to study these general energies via a "change of variables". In this process, some key tools arise, and I wish to tell you about them.

There are three tools:

1. A Lower semi-continuous energies on $\mathbb{R}^n$ with respect to measures μ:

$$\int |\nabla u|^p \, d\mu$$

2. Alberti Representations for μ: Decompositions of μ in the form

$$\mu(A) = \int \nu_\gamma(A) dP_\gamma,$$

   where P is some measure on curves and ν_γ are measures supported on such curves.

3. Upper gradients for Lipschitz functions $f : \mathbb{R}^n \to \mathbb{R}$: Non-negative Borel functions g that satisfy

$$|f(\gamma(0)) - f(\gamma(1))| \leq \int_\gamma g ds.$$

These concepts should look entirely different! However, there are ways to pass between them:

1. The Rainwater Lemma

2. Approximations of Lipschitz functions

3. Cone-null sets

Finally, there is a notion of a tangent space, the decomposability bundle of Alberti and Marchese $T_\mu(x)$, associated to a measure μ on $\mathbb{R}^n$. A very deep theorem of De Philippis and Rindler states:

Theorem 2.1.1

Let $\mu = \mu_s + \mu_a$ be the Lebesgue decomposition of μ with respect to the Lebesgue measure $\mathcal{L}$. Then for μ_s-a.e. $x \in \mathbb{R}^n$ we have $\dim(T_\mu(x)) \leq n-1$.

The goal of these lectures is to give a rigorous definition of the three objects above, and then to explain the two tools - with the aim of sketching how these tools come together with the result of de Philippis and Rindler. We will emphasize the role of lower-semicontinuity in energies, and aim to give a unifying approach to both Analysis on Fractals and Analysis on Metric spaces.

2.2 Energies

Definition 2.2.1: Dirichlet structure

A local Dirichlet structure is a touple $(X, \tau, \mathcal{X}, \mu, \mathcal{E})$ that satisfies

1. (X, τ) is a toplogical space and $(X, \mathcal{X}, \mu)$ is a measure space with $\mathcal{X}$ being the Borel σ-algebral.

2. $(u, v) \to \mathcal{E}(u, v)$ is a bilinear, positive definite symmetric form defined for some $u, v \in \mathcal{F}_2 \subset L^2(X, \mu)$.

3. $\mathcal{F}_2$ is a dense subset of $L^2(X)$.

4. **Closedness:** $\mathcal{F}_2$ is a Hilbert space when equipped with the inner product $\mathcal{E}_1(u, v) = \langle u, v\rangle + \mathcal{E}(u, v)$.

5. **Local:** If X is additionally a topological space, we say that a Dirichlet structure is strongly local, if for every open set A and if $g|_A = c$, then
$$\Gamma_2\langle g\rangle(A) = 0.$$

6. **Energy measure:** For every $f \in \mathcal{F}_2$ there exists a measure $\Gamma_2\langle f\rangle$ for which
$$\mathcal{E}(f, f) = \Gamma_2\langle f\rangle,$$
and
$$\Gamma_2\langle f + g\rangle(A)^{\frac{1}{2}} \leq \Gamma_2\langle f\rangle(A)^{\frac{1}{2}} + \Gamma_2\langle g\rangle(A)^{\frac{1}{2}}$$

7. **Contraction:** If $g \in \mathrm{Lip}(\mathbb{R})$ and $g(0) = 0$, then for every $f \in \mathcal{F}$ we have $g \circ f \in \mathcal{F}$ and
$$d\Gamma_2\langle g \circ f\rangle \leq \mathrm{LIP}[g] d\Gamma_2\langle f\rangle.$$

Examples include:

1. $(\mathbb{R}^n, \mathcal{B}, \mathcal{L}, \mathcal{E})$, where $\mathcal{B}$ is the Borel σ-alebra, $\mathcal{L}$ is Lebesgue measure and

$$\mathcal{E}(u,v) = \int_{\mathbb{R}^n} \langle \nabla u, \nabla v \rangle d\mathcal{L}$$

and $\mathcal{F}_2 = W^{1,2}(\mathbb{R}^n)$, with

$$\Gamma_2\langle f \rangle(A) = \int_A \langle \nabla u, \nabla v \rangle d\mathcal{L}.$$

This definition naturally extends to domains Ω and Riemannian manifolds M. Care must be taken in defining $\mathcal{F}_2$ and there may be multiple options for it on a given space, corresponding to different ways one treats boundaries/infinity.

2. If (X, d, μ) is a metric measure space, and $G_n = (V_n, E_n)$ is some sequence of bounded valence incidence graphs of positive measure Borel subsets of X, then consider

$$\mathcal{E}_n(f,g) = \sum_{a \sim b \in V_n} (f_u - f_v)(g_u - g_v)\sigma_n,$$

where $\sigma_n > 0$ is some chosen positive number and $f_A = \frac{1}{\mu(A)} \int_A f\mu$.
Let $\mathcal{E}$ be a Γ-*limit* of the energies $\mathcal{E}_n$. In many settings, one can show that $\mathcal{E}$ defines a Dirichlet structure. However, to get that $\mathcal{F}_2$ is dense, this often requires a well-chosen sequence σ_n and some structural properties - e.g. symmetry, to ensure that the energies do not blow-up or degenerate for a sufficiently large class of functions. This approach is the beginning of a field called *Analysis on fractals*. There are several wide open conjectures in this field, and the energies thus obtained remain poorly understood - even for many explicit fractals.

3. If (X, d, μ) is a metric measure space, we call g an upper gradient of $f \in L^2(X)$ if there is a measure zero set E s.t.

$$|f(\gamma(a)) - f(\gamma(b))| \leq \int_\gamma g ds,$$

for every rectifiable curve $\gamma : [a, b] \to X$ in X with $\gamma(a), \gamma(b) \notin E$. Let $N^{1,2}(X)$ be the collection of functions $f \in L^2(X)$ with some upper gradient $g \in L^2(X)$. A function g_f is called a minimal p-weak upper gradient for f if

a) **Minimality:** For every upper gradient $g \in L^2(X)$ of f, one has $g_f(x) \le g(x)$ for μ-a.e. $x \in X$.
b) **Weak upper gradient:** There exists a function $h \in L^2(X)$ s.t. $g_f + \epsilon h$ is an upper gradient for f for every $\epsilon > 0$.

(This definition is a bit condensed and avoids the use of modulus, which is customarily discussed at this juncture.) It is not hard to see from this definition, that a weak upper gradient is unique if it exists. An interested reader can verify as an exercise that such a weak upper gradient always exists.

Define

$$\mathcal{E}_2(f) = \int_X g_f^2.$$

This defines an energy on $N^{1,2}(X)$, and one can show that $N^{1,2}(X)$ defines a Banach space (after relaxing the definition by L^2 representatives). However, the energy doesn't always come with a bilinear form. This related setting is the starting point of *Analysis on Metric spaces*. This field is quite well understood and many manifold techniques extend here.

Despite there being two different settings for extending analysis from Euclidean spaces, a key property in both is lower semicontinuity.

Lemma 2.2.1

Let (X, d, μ) be a metric measure space, and $f_i \to f \in L^2(X)$. Then

$$\mathcal{E}_2(f) \le \liminf_{i\to\infty} \mathcal{E}_2(f_i).$$

Proof. We can assume that the limit inferior is a limit. We can further assume by passing to a subsequence that f_i converge to f outside a set N of measure zero. Let g_i be the upper gradients of f_i, let E_i be measure zero subsets in the definition. By taking convex combinations and Mazur, we may assume that g_i converges to some function h in $L^2(X)$ and $||h||_{L^2}^2 \leq \liminf_{i\to\infty} \mathcal{E}_2(f_i)$. Assume $||g_{f_i} - g||_{L^2(X)} \leq 4^{-i}$ by passing to a subsequence. Let $\epsilon > 0$ $g_\epsilon = h + \epsilon \sum_{i=1}^\infty 2^i |g_{f_i} - h|$.

Now, let $E = \bigcup_{i=1}^\infty E_i \cup N$ and h be given. We claim that g is an upper gradient for f. Let $\gamma : [a,b] \to X$ be a rectifiable curve with $\gamma(a), \gamma(b) \notin E$. Assume without loss of generality $\int_\gamma g_\epsilon ds < \infty$. Then, $|\int_\gamma g_i ds - \int_\gamma h ds| \leq 2^{-i} \int_\gamma g_\epsilon ds \to 0$ with $i \to \infty$. Thus, since f_i converges pointwise outside the set E, and g_i are upper gradients, we get

$$\begin{aligned}
|f(\gamma(a)) - f(\gamma(b))| &= \lim_{i\to\infty} |f_i(\gamma(a)) - f_i(\gamma(b))| \\
&\leq \liminf_{i\to\infty} \int_\gamma g_i ds \\
&\leq \liminf_{i\to\infty} \int_\gamma h ds \\
&\leq \liminf_{i\to\infty} \int_\gamma g_\epsilon ds.
\end{aligned}$$

Thus, g_ϵ is an upper gradient for f for every $\epsilon > 0$. This yields the claim, since then $\mathcal{E}(f) \leq \|g_\epsilon\|_{L^2}$ and sending $\epsilon \to 0$ yields

$$\mathcal{E}(f) \leq \|h\|_{L^2} \leq \liminf_{i\to\infty} \mathcal{E}_2(f_i),$$

as claimed. □

Lemma 2.2.2

Suppose that $(X, \mathcal{X}, \mu, \mathcal{E})$ is a Dirichlet structure, and that $\mathcal{F}_2$ is separable, then $\mathcal{E}_2(f) = \mathcal{E}_2(f, f)$ is lower semi-continuous. I.e. if

$f_i \to f \in L^2(X)$ and $f_i \in \mathcal{F}_2$, then

$$\mathcal{E}_2(f) = \liminf_{i\to\infty} \mathcal{E}_2(f_i).$$

Proof. Separable Hilbert spaces are reflexive, and thus by Mazur's lemma, there are convex combinations $\tilde{f}_k = \sum_{i=N_k}^{M_k} \alpha_{k,i} f_i$ with $\sum_{i=N_k}^{M_k} \alpha_{k,i} = 1$, $\alpha_{k,i} \geq 0$ and $\tilde{f}_k$ converges in $\mathcal{F}_2$ with respect to $\mathcal{E}_1$. Since $\mathcal{F}_2 \subset L^2(X)$ is a Hilbert space when equipped with the norm $\mathcal{E}_1$, we have that $\tilde{f}_k$ converges also in $L^2(X)$, and this limit must equal f. But, the energies of the convex combinations satisfy

$$\mathcal{E}_2(f) = \lim_{i\to\infty} \mathcal{E}_2(\tilde{f}_i) \leq \liminf_{i\to\infty} \mathcal{E}_2(f_i).$$

□

Motivated by these, we consider a weaker structure as below.

Definition 2.2.2: Energy structure

An energy structure is a touple $(X, \mathcal{X}, \mu, \mathcal{E})$ that satisfies

1. $(X, \mathcal{X}, \mu)$ is a measure space.
2. $\mathcal{F}_2$ is a dense subset of $L^2(X)$, and $f \to \mathcal{E}(f)$ is a seminorm.
3. **Closedness:** $\mathcal{F}_2$ is a Banach space when equipped with the norm $\mathcal{E}_1(u) = \langle u, v\rangle + \mathcal{E}(u)$.
4. **Local:** If X is additionally a topological space, we say that a Dirichlet structure is strongly local, if for every open set A and if $g|_A = c$, then
$$\Gamma_2\langle g\rangle(A) = 0.$$

5. **Energy measure:** For every $f \in \mathcal{F}_2$ there exists a measure $\Gamma_2\langle f\rangle$ for which
$$\mathcal{E}(f, f) = \Gamma_2\langle f\rangle,$$
and
$$\Gamma_2\langle f+g\rangle(A)^{\frac{1}{2}} \leq \Gamma_2\langle f\rangle(A)^{\frac{1}{2}} + \Gamma_2\langle g\rangle(A)^{\frac{1}{2}}$$
6. **Contraction:** If $g \in \mathrm{Lip}(\mathbb{R})$ and $g(0) = 0$, then for every $f \in \mathcal{F}$ we have $g \circ f \in \mathcal{F}$ and
$$d\Gamma_2\langle g \circ f\rangle \leq \mathrm{LIP}[g]d\Gamma_2\langle f\rangle.$$
7. **Lower semicontinuity:** If $f_i \to f \in L^p(X)$ and $f_i \in \mathcal{F}_2$, then
$$\mathcal{E}(f) \leq \liminf_{i\to\infty} \mathcal{E}(f_i),$$
where if the limit on the right is finite, we conclude that $f \in \mathcal{F}_2$.

This merges the Analysis on Metric spaces and Dirichlet form settings. Now, a complicated open problem is to understand what sorts of Energy structures exist. As posed this problem has too many variables, the space X, its metric, its measure and the energy form. In these lectures, to get a basic feel for this issue, we remove a ton of this variability. Let $(X, d, \mu) = (\mathbb{R}^n, d, \mu)$ be the usual Euclidean space equipped with a Radon measure μ, and consider the energies

$$\mathcal{E}(f) = \int |\nabla f|^2 d\mu, \Gamma_2\langle f\rangle(A) = \int_A |\nabla f|^2 d\mu$$

for $f \in C^1(\mathbb{R}^n)$. We ask: *When does there exist an energy structure $\mathcal{E}$ s.t. $\mathcal{E}$ agrees with the above for C^1 functions?* We call μ a 2-regular measure, if such an extension exists. Our main theorem, which we aim to explain is the following.

Theorem 2.2.3

If μ is 2-regular, then $\mu \ll \mathcal{L}$.

2.3 Special case: one dimension

We will prove Theorem 2.2.3 in the case of $n = 1$, since the proof illustrates already some useful ideas. This proof is quite classical, and has been used in many arguments: in proving the one dimensional version of the Energy Image Density property for Dirichlet forms, and in constructing functions that fail to be differentiable on zero measure sets.

Lemma 2.3.1

Let $L \geq 1$. Assume $K \subset \mathbb{R}$ is a closed set that satisfies $\mathcal{L}(K) = 0$, then for every $f \in C^1(\mathbb{R})$ with $\sup_{x\in\mathbb{R}} |f'(x)| \leq L$ there exists a sequence $f_i \in C^1(\mathbb{R})$ with

1. $\sup_{x\in\mathbb{R}} |f_i'(x)| \leq L$ for all $i \in \mathbb{N}$
2. $f_i \to f$ pointwise
3. $f_i'(x) = 0$ for every $x \in K$.

Proof. Let $\psi_i(x) = \max(id(x, K), 1)$. Let $g_i = f'\psi_i$. One shows that $g_i(x) \to f'(x)$ for every $x \notin K$. Indeed, if i is so large that $d(x, K) > 1/i$, then $g_i(x) = f'(x)$ since $\psi_i(x) = 0$. One can also check that $\sup_{x\in\mathbb{R}} |g_i(x)| \leq \sup_{x\in\mathbb{R}} |f'(x)| = L$ since $\psi_i(x) \in [0, 1]$ for all $x \in \mathbb{R}^n$.

Now, let

$$f_i(x) = f(0) + \int_0^x g_i(t)dt.$$

Dominated convergence theorem for integrals shows that $f_i \to f$ pointwise. Further $f_i'(x) = g_i(x) = 0$ for $x \in K$ by the fundamental theorem of calculus. □

Proof of Theorem 2.2.3 for $n = 1$. Suppose that K is a compact set s.t. $\mathcal{L}(K) = 0$. Let O be an arbitrary open set containing K. Let $f(x) = x$. With Lemma 2.3.1 we get a sequence of 1-Lipschitz f_i approximating f

with $f_i'(x) = 0$ for $x \in K$. Then, by the lower semi-continuity

$$\begin{aligned}\Gamma_2\langle f\rangle(K) &\le \Gamma_2\langle f\rangle(O)\\ &\le \liminf_{k\to\infty}\Gamma_2\langle f_k\rangle(O)\\ &\le \liminf_{k\to\infty}\Gamma_2\langle f_k\rangle(O\setminus K)\\ &\le \liminf_{k\to\infty}\mathcal{L}(O\setminus K) = \mathcal{L}(O\setminus K).\end{aligned}$$

Since O is arbitrary, we get

$$\Gamma_2\langle f_v\rangle(K) = \mu(K) = 0.$$

Thus $\mu \ll \mathcal{L}$. □

Chapter 3

Lecture 2: Upper gradients and Approximations

3.1 Warmup

In proving Theorem 2.2.3 for $n \geq 3$, we will need a much more involved argument. This argument uses the notion of cone null sets. These sets generalize measure zero sets as we used them before.

Let $v \in \mathbb{R}^n$ be a unit vector and $\theta \in (0, \pi/2)$. A cone is a set

$$C(v, \theta) := \{w : \langle w, v \rangle \geq \cos(\theta)|w|\}.$$

Definition 3.1.1

A compact set $K \subset \mathbb{R}^n$ is $C(v, \theta)$-cone null, if for every absolutely continuous curve $\gamma : [0, 1] \to \mathbb{R}^n$, and a.e. $t \in [0, 1]$ one of the following two true:

1. $\gamma(t) \notin K$
2. $\gamma'(t) = 0$ or $\gamma'(t) \notin C(v, \theta) \cup C(-v, \theta)$.

Example: Let $K = [0, 1] \times C$, where C is a cantor set, $v = (0, 1)$ and $\theta = \pi/4$. Then K is $C(v, \theta)$-null.

Our goal will be to show the following.

Theorem 3.1.1

If μ is 2-regular and K is C-cone null, then $\mu(K) = 0$.

This theorem can be thought of as a type of absolute continuity statement. However, to connect it with the absolute continuity with respect to $\mathcal{L}$ we will need to add a few more steps later.

The proof of Theorem 3.1.1 consists of two ideas: a higher dimensional version of the 1-d approximation that we already saw, and a compactness argument. A new tool of an upper gradient will appear.

3.2 A compactness argument

We start off by introducing a compactness argument from the study of modulus. While we do not need modulus in these lectures, this discussion will be instructive, will make the ideas a bit more transparent, and is intricately connected to the task of approximating functions. Indeed, historically one of the definitions of a Sobolev space in metric spaces involved the notion of modulus. Lipschitz functions and their approximations on the other hand have many parallels with approximating Sobolev functions - and indeed the theory of Lipschitz functions can be seen as a $p = \infty$ extension of the Sobolev theory which is defined for finite p.

Let $E, F \subset [0,1]^d$ be two disjoint, non-empty, compact sets. Let $\Gamma(E, F)$ be the family of all rectifiable curves connecting E to F. Consider the following optimization problem:

$$\mathrm{Mod}_2(E, F) = \inf\{\int_{[0,1]^2} \rho^2 d\mathcal{L} : \int_\gamma \rho ds \geq 1, \gamma \in \Gamma(E, F)\},$$

where the infimum is taken over all non-negative Borel functions ρ that satisfy the constraint.

One may ask: Can we take ρ to be continuous? With this additional restriction, we obtain $\text{Mod}_2^c(E, F)$, and it is clear that $\text{Mod}_2(E, F) \leq \text{Mod}_2^c(E, F)$. We will show equality.

Lemma 3.2.1

Let $n \geq 2$. Let $E, F \subset [0, 1]^n$ be two disjoint non-empty compact sets. Then

$$\text{Mod}_2(E, F) = \text{Mod}_2^c(E, F).$$

Proof. Let $\epsilon > 0$, and assume $\text{Mod}_2(E, F) < \infty$. Choose a non-negative Borel function ρ_b s.t. $\int_\gamma \rho_b ds \geq 1$ for all $\gamma \in \Gamma(E, F)$ and

$$\int \rho_b^2 d\mathcal{L} \leq \text{Mod}_2(E, F) + \epsilon/2.$$

By Vitali-Caratheodory, there exists a lower semicontinuous ρ_{lc} s.t. $\rho_{lc} \geq \rho$ and

$$\int \rho_{lc}^2 d\mathcal{L} \leq \text{Mod}_2(E, F) + \epsilon.$$

W have $\int_\gamma \rho_{lc} ds \geq 1$ since $\rho_b \leq \rho_{lc}$.

Further, lets consider $\rho_g(x) = \rho_{lc}(x) + \eta$.

Every non-negative lower semicontinuous function is an increasing limit of continuous non-negative functions: $\rho_{i,c} \nearrow \rho_{lc}$.

Define $\rho_i = \rho_{i,c} + \eta$, which is continuous. Let

$$a_i = \inf_{\gamma \in \Gamma(E,F)} \int_\gamma \rho_i ds.$$

We will show at the end of the proof that $\lim_{i\to\infty} a_i = 1$ from which the claim follows, since then

$$\text{Mod}_2^c(E, F) \leq \lim_{i\to\infty} \frac{\int \rho_i^2 d\mathcal{L}}{a_i^2} = \int (\rho_{lc} + \eta)^2 d\mathcal{L}.$$

This holds for every $\eta > 0$ so sending $\eta \to 0$ yields

$$\mathrm{Mod}_2^c(E,F) \le \mathrm{Mod}_2(E,F) + \epsilon,$$

and since $\epsilon > 0$ is arbitrary the claim follows.

Now, we are left to show $\lim_{i\to\infty} a_i = 1$. This is the really important idea and step in this proof. Suppose not, then there exists a $\delta > 0$ sequence of $\gamma_i \in \Gamma(E,F)$ s.t.

$$\int_{\gamma_i} \rho_i ds \le 1 - \delta.$$

Since $\rho_i \ge \eta$, we have $\mathrm{Len}(\rho_i) \le \frac{1}{\eta}$.

Thus, by Arzelá-Ascoli there exists a subsequence of γ_i that converges uniformly to a curve γ. But, then for every $j \in \mathbb{N}$:

$$\int_\gamma \rho_j ds \le \liminf_{k\to\infty} \int_{\gamma_k} \rho_j ds \le \liminf_{k\to\infty} \int_{\gamma_k} \rho_k ds \le 1 - \delta.$$

Sending $j \to \infty$ and using monotone convergence gives

$$\int_\gamma \rho_{lc} + \eta ds \le 1 - \delta,$$

but this is a contradiction since $\int_\gamma \rho_{lc} ds \ge 1$. □

3.3 Approximation and upper gradients

In the modulus calculation, we looked at the inequality

$$1 \le \int_\gamma g ds.$$

The upper gradient inequality is very similar to this, but with a variable left hand side. This observation has been key to many recent results in metric spaces, and illustrates a key concept: modulus and function spaces are always closely tied.

Definition 3.3.1

A function g is an upper gradient for $f : \mathbb{R}^n \to \mathbb{R}$ if

$$|f(\gamma(0)) - f(\gamma(1))| \leq \int_\gamma g ds$$

for every rectifiable curve $\gamma : [0, 1] \to \mathbb{R}^n$.

If $C = C(v, \theta)$, denote $-C = C(-v, \theta)$ the opposite cone of C.

Lemma 3.3.1

Assume that K is a compact set, $f \in C^1(\mathbb{R}^n)$ and K is $C = C(v, \theta)$-cone null. Then, $g = |\nabla f| 1_{\mathbb{R}^n \setminus K} + \sup_{|w|=1, w \notin C \cup -C} |\langle \nabla f, w \rangle| 1_K$. Moreover g is lower semicontinuous.

Proof. If $\gamma : [0, 1] \to \mathbb{R}^n$ is an absolutely continuous curve, then

$$f \circ \gamma'(t) = \langle \gamma', \nabla f \rangle.$$

If $\gamma(t) \notin K$, then $|f \circ \gamma'(t)| \leq g(\gamma(t))|\gamma'(t)|$ by Cauchy's inequality. If $\gamma'(t) \notin C \cup -C$, then also $|f \circ \gamma'(t)| \leq g(\gamma(t))|\gamma'(t)|$ since $g(\gamma(t)) = \sup_{|w|=1, w \notin C} |\langle \nabla f(\gamma(t), w \rangle|$ if $\gamma(t) \in K$.

By cone nullity, these two cases cover a.e. $t \in [0, 1]$ and we get

$$\begin{aligned} |f(\gamma(1)) - f(\gamma(0))| &\leq \int_0^1 |(f \circ \gamma)'(t)| dt \\ &\leq \int_0^1 g(\gamma(t))|\gamma'(t)| dt \\ &= \int_\gamma g ds. \end{aligned}$$

Thus, g is an upper gradient. □

Lower semi-continuous upper gradients are good for the following reason.

Theorem 3.3.2

Let $\epsilon > 0$ and suppose that a bounded function $f \in C(\mathbb{R}^n)$ has a lower semi-continuous upper gradient g. There exists a sequence $f_i \in \mathrm{Lip}(\mathbb{R}^n)$ and continuous functions g_i s.t. g_i is an upper gradient for f_i and $f_i \to f$ uniformly. Moreover, if g is bounded by L, we can choose g_i to be bounded by $L + \epsilon$.

Proof. It may be helpful to carefully think about the Modulus proof above and try and match these steps with it.

First, let $\tilde{g}_i \nearrow g$ be the sequence of continuous non-negative functions converging to g. Let

$$g_i = \tilde{g}_i + \epsilon.$$

Now, let

$$f_i(x) = \inf\{f(\gamma(0)) + \int_\gamma g_i ds : \gamma(1) = x\},$$

where the infimum is taken over all absolutely continuous γ.

First, g_i is an upper gradient for f_i. Indeed, if $\sigma : [0,1] \to \mathbb{R}^n$ is any curve, and $\gamma : [0,1] \to \mathbb{R}^n$ is any curve curve with $\gamma(1) = \sigma(0)$, then the concatenation $\sigma \star \gamma$ of γ and σ connects to $\sigma(1)$. Thus

$$\begin{aligned} f_i(\sigma(1)) &\leq f(\sigma \star \gamma(0)) + \int_{\sigma\star\gamma} g_i ds \\ &\leq f(\gamma(0) + \int_\gamma g_i ds + \int_\sigma g_i ds. \end{aligned}$$

Taking an infimum over γ, we get

$$f_i(\sigma(1)) - f_i(\sigma(0)) \leq \int_\sigma g_i ds.$$

By reversing the path σ, one gets the opposite inequality, and thus

$$|f_i(\sigma(1)) - f_i(\sigma(0))| \leq \int_\sigma g_i ds. \tag{3.1}$$

That is, g_i is the upper gradient of f_i.

Further, g_i are bounded by $L = \sup_{x\in\mathbb{R}^n} g(x) + \epsilon$, and if σ is the straight line segment between x, y, then the upper gradient estimate (3.1) yields

$$|f_i(x) - f_i(y)| \leq L|x - y|,$$

i,e. f_i is L-Lipschitz.

We are left to show that $f_i \longrightarrow f$ pointwise. Suppose this is not the case for some $x \in \mathbb{R}^n$. Clearly $f_i(x) \leq f(x)$, since we can take a constant path γ in the definition of f_i. Thus, if the claim fails, there is some $\delta > 0$ s.t.

$$f_i(x) \leq f(x) - \delta.$$

By definition of f_i, there exists a γ_i s.t.

$$f(\gamma_i(0)) + \int_{\gamma_i} g_i ds \leq f(\gamma_i(1)) - \delta$$

and $\gamma_i(1) = x$

Now, since f is bounded, there is a constant M s.t. $|f(y)| \leq M$ for all $y \in \mathbb{R}^n$, and thus

$$\int_{\gamma_i} g_i ds \leq f(\gamma_i(1)) - f(\gamma_i(0)) - \delta \leq 2M.$$

Since $g_i \geq \epsilon$, we have

$$\text{Len}(\gamma_i) \leq \frac{2M}{\epsilon}$$

Thus, by Arzela-Ascoli we can find a subsequence of γ_i that converges uniformly to some γ. Then

$$\begin{aligned}\int_\gamma g_j ds &\leq \liminf_{k\to\infty} \int_{\gamma_k} g_j ds \\ &\leq \liminf_{k\to\infty} \int_{\gamma_k} g_i ds \\ &\leq f(\gamma(1)) - f(\gamma(0)) - \delta.\end{aligned}$$

Sending $j \to \infty$, we get

$$\int_\gamma g+\epsilon ds = \lim_{j\to\infty} \int_\gamma g_j ds \leq f(\gamma(1))-f(\gamma(0))-\delta \leq |f(\gamma(1))-f(\gamma(0))|-\delta,$$

which is a contradiction to the definition of an upper gradient.

□

3.4 Proof of cone-nullity

Proof of Theorem 3.1.1. Let μ be a 2-regular measure and K be a compact set in $\mathbb{R}^n$ that is $C = C(v,\theta)$-null for some unit vector $v \in \mathbb{R}^n$ and $\theta \in (0, \pi/2)$.

Consider $f(x) = \langle x, v\rangle$. Then, by Lemma 3.3.1 we have that $g = 1_{\mathbb{R}^n\setminus K} + \cos(\theta)1_K$ is an upper gradient for f. This function g is lower semi-continuous, and bounded by 1. Thus, by Theorem 3.1.1 we have a sequence f_i of 2-Lipschitz functions which converge pointwise to f and with continuous upper gradients g_i s.t. $g_i \nearrow g + (1 - \cos(\theta))/2$. But, then

$$\begin{aligned}\Gamma_2\langle f_i\rangle(K) &= \int_K |\nabla f_i|^2 d\mu \\ &\leq \int_K \cos(\theta) + (1 - \cos(\theta)/2) d\mu \\ &= (\cos(\theta) + (1 - \cos(\theta)/2)\mu(K).\end{aligned}$$

But,

$$\Gamma_2\langle f\rangle(K) = \int_K |\nabla f|^2 d\mu = \mu(K),$$

which is a contradiction, since $\cos(\theta) + (1 - \cos(\theta)/2) < 1$, and thus

$$\liminf_{i\to\infty} \Gamma_2\langle f_i\rangle(K) < \Gamma_2\langle f\rangle(K).$$

In this proof, we took a small shortcut: the functions f_i are not C^1. This is not serious, and could be fixed by taking convolutions. We leave this inessential detail out to emphasize the main ideas. □

Chapter 4

Lecture 3: Rainwater and Alberti Representations

4.1 Cone nullity

We wish to now move from cone absolute continuity to absolute continuity with respect to Lebesgue. That is, we wish to show the following.

A measure μ on $\mathbb{R}^n$ is said to be absolutely continuous with respect to another measure ν if $\nu(A) = 0$ implies $\mu(A) = 0$. This is equivalent to finding some non-negative measurable function f s.t.

$$\mu(A) = \int_A f d\nu.$$

Theorem 4.1.1

If μ is a locally finite Radon measure on $\mathbb{R}^n$, and

$$\mu(K) = 0$$

for every compact K that that is C-cone null for some cone C, then $\mu \ll \mathcal{L}$.

If we succeed, then we have a proof of our main objective.

Proof of Theorem 2.2.3 for $n \geq 3$. By Theorem 3.1.1, we have that

$$\mu(K) = 0$$

for every compact K that that is C-cone null for some cone C. Thus, $\mu \ll \mathcal{L}$ by Theorem 4.1.1.

□

4.2 Alberti representations

The proof of Theorem 4.1.1 goes via a notion of a tangent space T_μ associated to every measure μ. To define this, we need to study Alberti representations and connect them to our notion from before. The key tool in today's lecture is the Rainwater lemma.

Definition 4.2.1

An Alberti representation is a pair (P, ν) of a measure P on curves $\Gamma(\mathbb{R}^n)$ and a map $\gamma \to \nu_\gamma$, which associates to every curve a finite Borel measure on $[0, 1]$ s.t.

$$\gamma \to \nu_\gamma(\{t : \gamma(t) \in A\})$$

is Borel measurable for every Borel set A,

$$\mu(A) = \int \nu_\gamma(A) dP_\gamma,$$

and

$$\nu_\gamma \ll \mathcal{L}, \nu_\gamma(\{t : \gamma'(t) = 0\}) = 0.$$

At this juncture, and since this is a minicourse, we will declare that all measurability discussions will be skipped. These are extremely important, but are handled with some technical and somewhat straightforward ways. A good introduction to them is given by the woksr of Andrea Schioppa on the topic. Also my joint work with Elefterios Soultanis handle some of them.

Example: Let $\mu = 1_{\mathcal{L}}$, let P be the uniform measure on the curves $\gamma_t : [0,1] \to [0,1]^2$ given by $\gamma_t(s) = (t,s)$, where the uniform measure is on the parameter $t \in [0,1]$. Let ν_γ be Lebesgue measure on $[0,1]$. Then, by Fubini's theorem

$$\begin{aligned}\mu(A) &= \int_0^1 \int_0^1 1_A(s,t)dsdt \\ &= \int_0^1 \int_0^1 1_A(\gamma_t(s))dsdt \\ &= \int_0^1 \nu_\gamma(\{s : \gamma_t(s) \in A\})dt,\end{aligned}$$

and we see that (P,ν) is an Alberti representation. This is the first example to understand and illustrates how Alberti representations are closely connected to Fubini.

We say that an Alberti representation (P,ν) is in the direction of a cone C, if for P-a.e. γ and ν_γ-a.e. $t \in [0,1]$ we have $\gamma'(t) \in C$.

In the example above, the Alberti representation (P,ν) is in the direction $C = C(e_1,\theta)$ for any $\theta \in (0,\pi/2)$. Indeed, the Alberti representation is in the constant x direction. It is extremely rare for Alberti representations to be so nice, but sometimes - especially after blowing up - such a nice setting occurs.

Theorem 4.2.1

If μ is a measure that has an Alberti representation in the cone direction C, then $\mu(K) = 0$ for every C-cone null set K.

Proof. Let (P,ν) be the Alberti representation in the cone direction C and let K be C-cone null. Then for P-a.e. $\gamma : [0,1] \to \mathbb{R}^n$ we have for a.e. $t \in [0,1]$ that $\gamma(t) \notin K$ or $\gamma'(t) \notin C$ or $\gamma'(t) = 0$. Thus, if $t \in \gamma^{-1}(K)$ then $\gamma'(t) = 0$ or $\gamma'(t) \notin C \cup -C$. Since for ν_γ-a.e. $t \in [0,1]$ we have

$\gamma'(t) \in C$, we get that up to a null-set, where γ' is not defined

$$\gamma^{-1}(K) \subset \{t : \gamma'(t) = 0\} \cup \{t : \gamma'(t) \notin C \cup -C\}$$

Thus, $\nu_\gamma(\gamma^{-1}(K)) = 0$

$$\nu_\gamma(\gamma^{-1}(K)) = 0$$

and

$$\mu(K) = \int \nu_\gamma(K) dP_\gamma = 0.$$

□

Our task is to show the converse to this.

Theorem 4.2.2

If $\mu(K) = 0$ for every compact set K which is C-cone null, then μ has an Alberti representation in the direction of C.

4.3 Rainwater lemma

To find Alberti representation in the proof of Theorem 4.2.2, we will need to apply a tool called Rainwaters Lemma. Fix a cone C. First, let $K_{pre} = \{\nu_\gamma\}$, where we take all measures ν_γ where γ is a rectifiable curve and ν_γ satisfies $\nu_\gamma \ll \mathcal{L}$ and $\nu_\gamma(\{t : \gamma'(t) \notin C\}) = 0$. An Alberti representation is a measure supported on K_{pre}, or in other words on element in the convex hull K of K_{pre} in $M(\mathbb{R}^n)$ - the space of measures. We want to characterize the measures which are equal to some measure in K. We will pretend that K is compact, and weaken equality to $\mu \ll \nu$ for some $\nu \in K$. With a number of technical modifications on the collection G it can be ensured that a modification of K is compact, but this white lie will simplify the tretment. Also, absolute continuity can be upgraded to having an Alberti representation.

A measure μ on $\mathbb{R}^n$ is said to be singular with respect to ν if there exists a measurable set E s.t. $\mu(E) = \mu(\mathbb{R}^n)$ and $\nu(E) = 0$. We write this as $\mu \perp \nu$

Let μ be any measure on $\mathbb{R}^n$. The Lebesgue decomposition says the following.

Theorem 4.3.1

There exists a decomposition

$$\mu = \mu_s + \mu_a,$$

where $\mu_a \ll \mathcal{L}$ and μ_s is singular with respect to Lebesgue, i.e. there is some measurable set $E \subset \mathbb{R}^n$ s.t. $\mathcal{L}(E) = 0$ and $\mu_s(E) = \mu_s(\mathbb{R}^n)$.

A generalization of this to a family of measures is the following.

Theorem 4.3.2

Let μ be a measure in $\mathbb{R}^n$ and K a compact collection of measures in $\mathbb{R}^n$. Then, there exists a decomposition

$$\mu = \mu_a + \mu_s$$

where $\mu_a \ll \nu$ for some $\nu \in K$ and there exists a measurable set N s.t. $\mu_s(N) = \mu_s(\mathbb{R}^n)$ and $\lambda(N) = 0$ for all $\lambda \in K$. In particular, $\mu_s \perp \lambda$ for all $\lambda \in K$.

The proof of this theorem is based on the mini-max principle in convex optimization.

Theorem 4.3.3

Let $G \subset X$ be a convex subset of a topological vector space X and $K \subset Y$ a compact convex subset of a topological vector space Y. Assume that

$$\Phi : G \times K \to \mathbb{R}$$

satisfies:

1. $\Phi(\cdot, y)$ is convex and lower semicontinuous for every $y \in K$, and
2. $\Phi(x, \cdot)$ is concave and upper semicontinuous for every $x \in G$.

Then,

$$\inf_{x\in G}\sup_{y\in K}\Phi(x,y)=\sup_{y\in K}\inf_{x\in G}\Phi(x,y)$$

This is a magical property, whose proof is intimately tied with convex optimization and linear programming (where it is often known as the Von Neumann minimax principle). In these settings, one often applies this principle to the Lagrangian. The proof is beautiful, see e.g. a paper by Komiya, *Elementary proof of Sion's minimax theorem.* I find it instructive to think of why the claim is not obvious, and why

$$\sup_{y\in K}\inf_{x\in G}\Phi(x,y)\leq\inf_{x\in G}\sup_{y\in K}\Phi(x,y).$$

When applied to the Lagrangian, this inequality is often called weak duality. The equality in Theorem 4.3.3 corresponds to strong duality, and the assumptions of convexity and concavity are key.

Let us see how to use this theorem to prove Theorem 4.3.2.

Proof of Theorem 4.3.2. Let $S=\sup\{\mu(E):\mu|_E\ll\nu,\text{ for some }\nu\in K\}$. There is a sequence E_i s.t. $\mu(E_i)\nearrow S$ maximizes this supremum and $\mu|_{E_i}\ll\nu_i$ with $\nu_i\in K$. Let $E=\bigcup E_i$. Then, $\nu=\sum_{k=1}^{\infty}2^{-i}\nu_i\in K$, since $\sum_{i=1}^{\infty}2^{-i}=1$, and since K is closed. It is then not hard to see that $\mu|_E\ll\nu$, and $\mu(E)=S$. Further $\mu|_{\mathbb{R}^n\setminus S}$ is singular with respect to every $\nu\in K$, since otherwise there would be some positive measure $A\subset\mathbb{R}^n\setminus S$ with $\mu(A)>0$ and $\mu|_A\ll\nu'$ for some $\nu'\in K$. But, then $\mu|_{A\cup E}\ll\frac{1}{2}(\nu+\nu')$, and $\mu(A\cup E)>S$, which a contradiction to the definition of S. We now let $\mu_a=\mu|_E$, and $\mu_s=\mu-\mu_a=\mu|_{\mathbb{R}^n\setminus E}$. We are now left to construct the set N for μ_s.

Lets now simplify notation and focus on $\mu=\mu_s$, and our aim is to find N. That is, we know that $\mu\perp\nu$ for every $\nu\in K$. (Note that this is actually the main case that we will need in the proof of Theorem 3.1.1).

Let K be the collection of measures equipped with the weak topology, and let $G\subset C(\mathbb{R}^n)$ be the collection of continuous functions $\rho:\mathbb{R}^n\to$

$[0,1] \subset \mathbb{R}$. Consider the functional:

$$\Phi_M(g,\nu) = \int (1-g)d\mu + \int g d\nu.$$

For every ν, we have $\nu \perp \mu$, and thus there is a set B s.t. $\mu(B) = \mu(\mathbb{R}^n)$ and $\nu(B) = 0$. Since continuous functions are dense, then there exists a sequence of continuous functions g_i s.t. $g_i \longrightarrow 1_B$ (where the convergence is μ-a.e. and ν-a.e.). Thus,

$$0 \leq \inf \Phi_M(g,\nu) \leq \lim_{i\to\infty} \Phi(g_i,\nu) = \int 1_{\mathbb{R}^n\setminus B} d\mu + \int 1_B d\nu = 0.$$

The functional Φ is affine in ν and ρ, and is continuous in ρ and ν. Thus

$$\inf_{g\in G} \sup_{\nu\in K} \Phi(g,\nu) = \sup_{\nu\in K} \inf_{g\in G} \Phi(g,\nu) = 0.$$

Thus, there exists a sequence of functions $g_M \in G$ for which

$$\int g_M d\nu + \int (1-g_M)d\mu = \Phi(g_M,\nu) \leq 2^{-M}$$

for all $\nu \in K$. Let $h = \sum_{M=1}^{\infty}(1-g_M)$, and let $N = \{x : h(x) < \infty\}$.

Now,

$$\int h d\mu = \sum_{i=1}^{\infty} \int (1-g_i)d\nu \leq 1,$$

and thus $\mu(N) = \mu(\mathbb{R}^n)$. On the other hand, if $x \in N$, then $1 - g_M \longrightarrow 0$, and $h_1(x) = \sum_{i=1}^{\infty} g_M(x) = \infty$. But, for every $\nu \in K$

$$\int h_1 d\nu \leq 1,$$

and thus $\nu(N) = 0$ for every $\nu \in K$, and the claim has been shown. □

4.4 Applying Rainwater to get Alberti representations

We now prove the existence of Alberti representations.

Proof. Let $\mathcal{K}$ be the collection of all measures ν s.t. ν has an Alberti representations in the cone direction C. Then, up to a serious but manageable compactness issue (see e.g. some arguments by Schioppa), one can use Rainwaters lemma 4.3.2 and get a secomposition

$$\mu = \mu_a + \mu_s,$$

and a measurable set, s.t.

1. μ_a has an Alberti representation in the cone direction C
2. $\mu_s(E) = \mu_s(\mathbb{R}^n)$ and $\nu(E) = 0$ for every $\nu \in \mathcal{K}$.

We argue that $\mu_s(E) = 0$. If not, then we could find some compact set $K \subset E$ s.t. $\mu_s(K) > 0$ and $\nu(E) = 0$. Now, let γ be arbitrary, and let $\nu_\gamma = \lambda|_{\gamma' \neq 0, \gamma' \in C}$. This defines an Alberti representation in the direction of C. Let $\nu(K) = \nu_\gamma(\gamma^{-1}(K))$. Since $\nu(K) = 0$, we get $\nu_\gamma(\gamma^{-1}(K)) = 0$, and this is precisely that for a.e. t we have $\gamma \notin K$, $\gamma' \notin C$ or $\gamma' = 0$. Thus K is C-cone null. But, this is a contradiction to $\mu(K) = \mu_s(K) > 0$, since μ was supposed to give measure zero to all cone null sets. □

Chapter 5

Lecture 4: Decomposability bundle and Proof of Main theorem

Let's collect some facts about 2-regular measures and all measures μ that we have shown

1. By Theorem 3.1.1. For every 2-regular measure we have $\mu(K) = 0$ for every K that is C-cone null for some C.
2. By Theorem 4.2.1: Every measure μ s.t. $\mu(K) = 0$ for every C-cone null set K has an Alberti representation in the cone direction C.

Combining these, we get:

Theorem 5.0.1

If μ is 2-regular, and if C is a cone, then μ has an Alberti representation in the cone direction of C.

Not that the above was easy, but after this things get even harder.

Definition 5.0.1

The decomposability bundle $T_\mu(x)$ is a mapping that associates to every $x \in \mathbb{R}^n$ a subspace of $\mathbb{R}^n$, so that

1. For every Alberti representation (P, ν_γ) of μ we have for P-a.e. γ and ν_γ-a.e. $t \in [0, 1]$ that

$$\gamma'(t) \in T_\mu(\gamma(t)).$$

2. If $T(x)$ is another such mapping, then

$$T_\mu(x) \subset T(x)$$

for μ-a.e. x.

Alberti-Marchese show that this exists, which its self takes some time to understand and accept. But, using this notion, we have:

Theorem 5.0.2

If μ is 2-regular, then $T_\mu(x) = \mathbb{R}^n$ for μ-a.e. $x \in \mathbb{R}^n$

Proof. By Theorem 5.0.1 we have that μ has an Alberti representation in any cone direction C, and thus $T_\mu(x) \cap C \neq \emptyset$ for μ-a.e. $x \in X$.

Thus $T_\mu(x)$ must contain all of $\mathbb{R}^n$, for μ-a.e. x and the claim follows. □

We can now complete the proof of Theorem 2.2.3.

Proof of Theorem 2.2.3 for $n > 1$. By Theorem 5.0.2, we have $\dim(T_\mu(x)) = n$ for μ-a.e. $x \in \mathbb{R}^n$. Thus, by 2.1.1 we have $\mu \ll \mathcal{L}$, as desired. □

Chapter 6

References

The contents of these lecture notes were from no single refence, but here is a collection of some very related reading.

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